Contents

INDEPENDENCE POLYNOMIALS AND EXTENDED VERTEX REDUCTION

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ABSTRACT. The independence polynomial of a graph is a polynomial whose coefficients number the independent sets of each size in that graph. This paper looks into methods of obtaining these polynomials for certain classes of graphs which prove too large to easily find the polynomial by traditional methods.

1. Introduction

Many times in the sciences, situations occur that involve networks of people or objects, all interrelated in various ways. Examples include the study of scheduling algorithms, the study of communication networks, molecular chemistry, and statistical physics. These networks can be modeled by constructs known as graphs which will be defined more rigorously later. Graph Theory, the study of these objects, is a rapidly expanding field in mathematics. Insights gained from its study provide invaluable knowledge about the underlying properties of things as diverse as computer systems and highway traffic. One relatively new aspect is the study of what is commonly referred to as the independence polynomial of a graph, a representation of the relations between the vertices in the graph. In this paper, I cover the basic definitions

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and theorems that lead up to the construction and manipulation of independence polynomials then proceed to share some results of my own in the pursuit of the generation of independence polynomials of graphs too difficult to compute by hand. I begin with definitions:

Definition 1.1. *An ordered pair G=(V,E) where V=V(G) is a finite set of vertices and E=E(G) is a set of edges, that is 2-element subsets of V, is called a graph.*

In this paper we will be strictly considering finite simple undirected graphs, and per convention they shall be simply referred to as graphs.

Definition 1.2. *Let G=(V,E) be a graph. Two vertices $v_1, v_2 \in V$ are said to be adjacent provided that $(v_1, v_2) \in E$.*

Definition 1.3. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be graphs. We say that H is a subgraph of G provided that $V(H) \subset V(G)$ and $E(H) \subset E(G)$.*

Definition 1.4. *Let $G(V, E)$ be a graph and $v \in V$. The neighborhood of v, $N\{v\}$, is the subgraph of G consisting of v and every $w \in V$ adjacent to v along with the corresponding edges.*

Definition 1.5. *Let G = (V,E) be a graph. A set of vertices $\{v_1, v_2, \ldots, v_k\}$ is called an independent set of size k provided that $\{v_a, v_b\} \notin E$ for any $a, b \in \{1, 2, \ldots, k\}$. That is to say it is a set of pairwise non-adjacent vertices.*

Definition 1.6. *Let G = (V,E) be a graph. G is said to be a k-clique if $|V| = k$ and $\{v_1, v_2\} \in E$ $\forall v_1, v_2 \in V$.*

Definition 1.7. [1] *Let G=(V,E) be a graph. The independence number* $\alpha(G)$ *is the cardinality of a maximum independent set of* G. *Let* $f_k = f_k(G)$ *be the number of independent sets of size* k *of* G. *The polynomial* $G(x) = \sum_{k=0}^{\alpha(G)} f_k x^k = f_0 + f_1 x + f_2 x^2 + \cdots + f_{\alpha(G)} x^{\alpha(G)}$ *is called the independence polynomial of* G.

Henceforth we shall use $G(x)$ to denote the independence polynomial of the graph G. Now, before progressing, we need a tool from outside of graph theory, namely that of solving recursion relations. For our puposes, only the following rudimentary theorem is necessary.

Theorem 1.8. [5] *Suppose a sequence of polynomials* $\{a_n\}_{n=0}^{\infty}$ *satisfies the recursion:*

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}$$

for $n \geq k \in \mathbb{N}$ *and* C_i *a polynomial* $\forall 1 \leq i \leq k$. *Then if* $r_1, r_2, \ldots, r_k$ *are the distinct roots of*

$$f(\lambda) = \lambda^k - C_1 \lambda^{k-1} - C_2 \lambda^{k-2} - \cdots - C_k$$

there are constants $b_1, b_2, \ldots, b_k$ *so that for all* n

$$a_n = b_1 r_1^n + b_2 r_2^n + \cdots + b_k r_k^n.$$

Example 1. Consider the sequence of polynomials $\varphi_n(x)$ where, for $n \geq 2$

$$\varphi_n(x) = \varphi_{n-1}(x) + x\varphi_{n-2}(x),$$

and

$\varphi_0 = 2$ and $\varphi_1 = 1$. Then we start by finding the roots of $\lambda^2 - \lambda - x$ which are

$$r_1 = \frac{1+\sqrt{1+4x}}{2}, r_2 = \frac{1-\sqrt{1+4x}}{2}$$

then by solving the first two cases in the sequence together,

$$\varphi_0(x) = b_1 + b_2$$

$$\varphi_1(x) = b_1 r_1 + b_2 r_2$$

we can ascertain the values of b_1, b_2 and obtain an explicit formula for $\varphi_n(x)$. For instance, consider the case when $\varphi_0(x) = 2$ and $\varphi_1(x) = 1$. Then we have

$$2 = b_1 + b_2$$

$$1 = b_1 r_1 + b_2 r_2$$

$$\implies b_1 = 2 - b_2$$

$$\implies 1 = 2r_2 + b_2(r_2 - r_1) = 2(\frac{1+\sqrt{1+4x}}{2}) + b_2(\frac{1-\sqrt{1+4x}}{2} - \frac{1+\sqrt{1+4x}}{2})$$

$$\implies 1 = 1 + \sqrt{1+4x} + b_2(-\sqrt{1+4x})$$

$$\implies b_2 = 1$$

$$\implies 2 = b_1 + 1$$

$$\implies b_1 = 1.$$

Now this is relevant to our ends as a very important tool, for the manipulation of independence polynomials involves recursion.

2. Generating Independence Polynomials

Theorem 2.1. [1] *Let G = (V,E) be a graph, and $v \in V$. Then*

$$G(x) = (G \backslash v)(x) + x(G \backslash N\{v\})(x),$$

where $(G \backslash v)$ is the subgraph formed by removing the vertex $v \in V$ and its incident edges from G, and $G \backslash N\{v\}$ is the subgraph formed by removing v, its neighbors, and their incident edges. This is known as the vertex reduction formula.

Now with this tool we can obtain recursion equations for many classes of graphs. Then we may apply our technique for solving recursions to generate explicit formulae for the independence polynomials of entire sequences of graphs. Begin by looking at a couple of the simplest graph sequences, the cycles and paths.

Definition 2.2. *Let $\mathbf{P}_n$ be the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\}$. Then $\mathbf{P}_n$ is called a path on n vertices. Let $\mathbf{C}_n$ be the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. Then $\mathbf{C}_n$ is called a cycle on n vertices.*

Figure 1. P_6

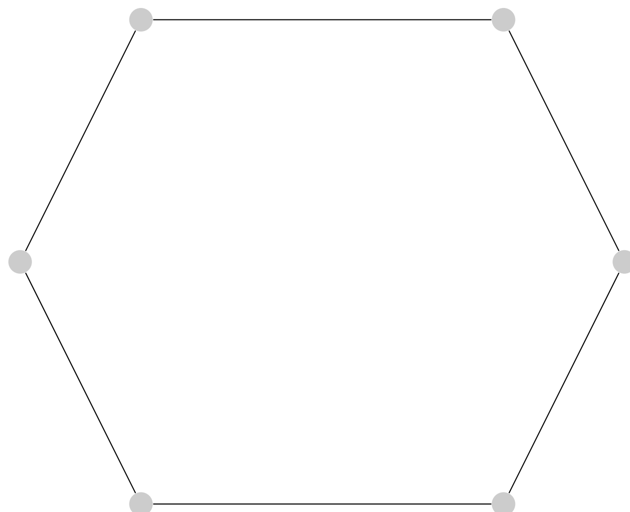

Figure 2. C_6

Theorem 2.3. *Recursions for Paths and Cycles.*

$P_n(x) = P_{n-1}(x) + xP_{n-2}(x)$

$P_0(x) = 1$

$P_1(x) = 1 + x$

$C_n(x) = P_{n-1}(x) + xP_{n-3}(x)$

By 1.8 we can model the recursion of $P_n(x)$ by

$$\lambda^2 = \lambda + x$$

$$\implies \lambda^2 - \lambda - x = 0$$

$$\implies \lambda = \frac{1 \pm \sqrt{1+4x}}{2}.$$

$$\lambda_1 = \frac{1 + \sqrt{1+4x}}{2}.$$

$$\lambda_2 = \frac{1 - \sqrt{1+4x}}{2}.$$

$$P_n(x) = f(x)\lambda_1^n + g(x)\lambda_2^n.$$

$$1 = f(x) + g(x) \implies f(x) = 1 - g(x).$$

$$1 + x = f(x)(\frac{1+\sqrt{1+4x}}{2}) + g(x)(\frac{1-\sqrt{1+4x}}{2}).$$

$$g(x) = \frac{1}{2} - \frac{1+2x}{2\sqrt{1+4x}}.$$

$$f(x) = \frac{1}{2} + \frac{1+2x}{2\sqrt{1+4x}}.$$

$$P_n(x) = (\frac{1}{2} + \frac{1+2x}{2\sqrt{1+4x}})(\frac{1+\sqrt{1+4x}}{2})^n + (\frac{1}{2} - \frac{1+2x}{2\sqrt{1+4x}})(\frac{1-\sqrt{1+4x}}{2})^n \implies$$

$$P_n(x) = \frac{1}{2^{n+1}}[((1+\sqrt{1+4x})^n + (1-\sqrt{1+4x})^n) + \frac{1+2x}{\sqrt{1+4x}}[(1+\sqrt{1+4x})^n - (1-\sqrt{1+4x})^n)]].$$

$$P_n(x) = \frac{1}{2}[(1)(C_n(x)) + (1+2x)(P_{n-2}(x))].$$

$$C_n(x) = 2P_n(x) - (1+2x)(P_{n-2}(x)).$$

Here are the values of P_n for the first 10 n:

$$1 + x$$

$$1 + 2x$$

$$1 + 3x + x^2$$

$$1 + 4x + 3x^2$$

$$1 + 5x + 6x^2 + x^3$$

$$1 + 6x + 10x^2 + 4x^3$$

$$1 + 7x + 15x^2 + 10x^3 + x^4$$

$$1 + 8x + 21x^2 + 20x^3 + 5x^4$$

$$1 + 9x + 28x^2 + 35x^3 + 15x^4 + x^5$$

$$1 + 10x + 36x^2 + 56x^3 + 35x^4 + 6x^5$$

Wingard[6] observed that the independence polynomials of the sequence of cycles mirrored that of the sequence of 3-cycles with a path attached. The next few endeavors were motivated by a desire to explore similar sequences of graphs.First we repeat the process with another sequence, T_n, formed by joining C_4 to an end vertex of an n-length path.

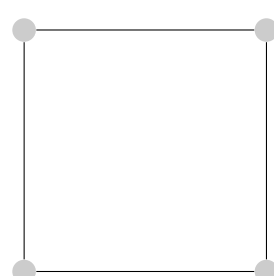

FIGURE 3. T_0

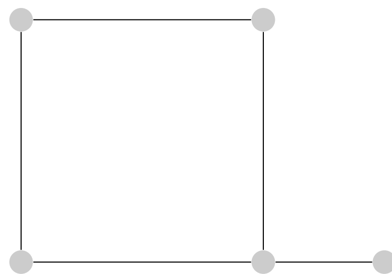

FIGURE 4. T_1

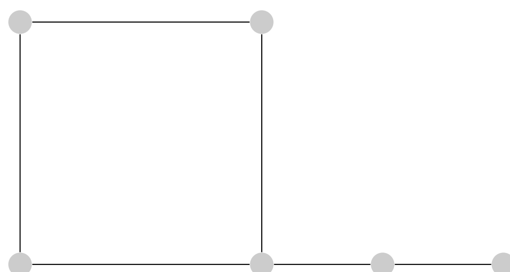

FIGURE 5. T_2

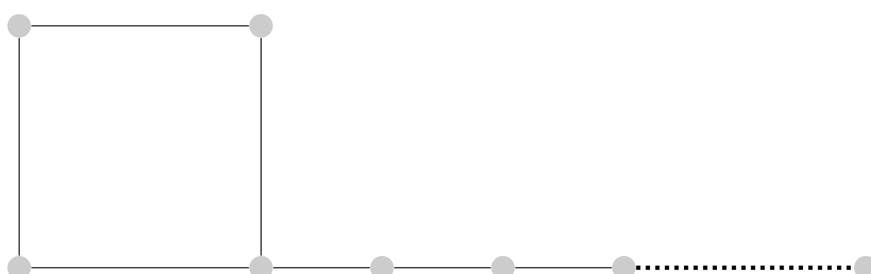

FIGURE 6. T_n

We derive the polynomials of the T_n sequence as above, using vertex reduction and 1.8.

$$T_n(x) = T_{n-1}(x) + xT_{n-2}(x)$$

$$T_0(x) = 1 + 4x + 2x^2$$

$$T_1(x) = 1 + 5x + 5x^2 + x^3$$

proceding as before we get the same lambdas:

$$T_n(x) = f(x)\lambda_1^n + g(x)\lambda_2^n.$$

$$1 + 4x + 2x^2 = f(x) + g(x) \implies f(x) = 1 + 4x + 2x^2 - g(x).$$

$$1 + 5x + 5x^2 + x^3 = f(x)\lambda_1 + g(x)\lambda_2.$$

$$g(x) = \frac{1 + 4x + 2x^2}{2} - \frac{1 + 6x + 8x^2 + 2x^3}{2\sqrt{1+4x}}.$$

$$f(x) = \frac{1 + 4x + 2x^2}{2} + \frac{1 + 6x + 8x^2 + 2x^3}{2\sqrt{1+4x}}.$$

$$\implies T_n(x) = \frac{1}{2^{n+1}}[(1 + 4x + 2x^2)(1 + \sqrt{1+4x})^n + (1 - \sqrt{1+4x})^n).$$

$$+ \frac{1 + 6x + 8x^2 + 2x^3}{\sqrt{1+4x}}[(1 + \sqrt{1+4x})^n - (1 - \sqrt{1+4x})^n)]].$$

$$\implies T_n(x) = \frac{1}{2}[(1 + 4x + 2x^2)(C_n(x)) + (1 + 6x + 8x^2 + 2x^3)(P_{n-2}(x))].$$

Definition 2.4. *A k-cliquepath, denoted by P_n^k, consists of a vertex set $V = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$ and edges joining any two vertices whose subscripts differ by 0 or 1.*

See the next page for examples, with $v's$ along the top, $w's$ along the bottom.

Definition 2.5. *A k-cliquecycle, denoted by C_n^k, consists of a vertex set $V = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$ and edges joining any two vertices whose subscripts differ by 0 or 1(mod n)*

Now we repeat the process for the graphs P_n^2.

For use in the vertex reduction, we display the graphs B_n.

Figure 7. P_1^2

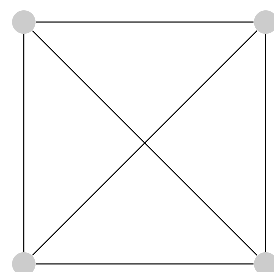

Figure 8. P_2^2

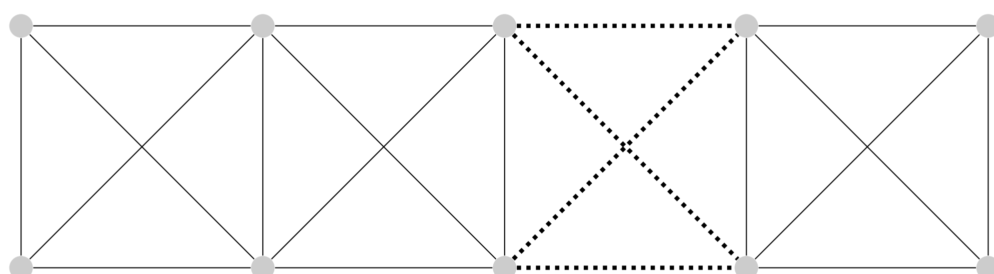

Figure 9. P_n^2

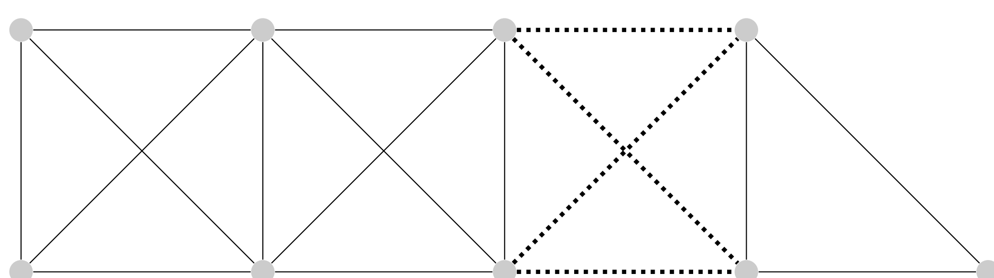

Figure 10. B_n

The program is as before, using vertex reduction to obtain a recursion which is solved by 1.8.

$$B_n(x) = P^2_{n-1}(x) + xP^2_{n-2}(x).$$

$$P^2_n(x) = B_n(x) + xP^2_{n-2}(x) \implies$$

$$P^2_n(x) = P^2_{n-1}(x) + 2xP^2_{n-2}(x).$$

$$P^2_0(x) = 1.$$

$$P^2_1(x) = 1 + 2x.$$

$$\lambda^2 = \lambda + 2x.$$

$$\lambda_1 = \frac{1+\sqrt{1+8x}}{2}.$$

$$\lambda_2 = \frac{1-\sqrt{1+8x}}{2}.$$

$$P^2_n(x) = f(x)\lambda_1^n + g(x)\lambda_2^n.$$

$$f(x) = \frac{1}{2} + \frac{1+4x}{2\sqrt{1+8x}}.$$

$$g(x) = \frac{1}{2} - \frac{1+4x}{2\sqrt{1+8x}}.$$

$$P^2_n(x) = \frac{1}{2^{n+1}}[(1)(1+\sqrt{1+8x})^n + (1-\sqrt{1+8x})^n) + \frac{1+4x}{\sqrt{1+8x}}[(1+\sqrt{1+8x})^n - (1-\sqrt{1+8x})^n)]].$$

$$P^2_n(x) = \frac{1}{2}[(1)(C^2_n(x)) + (1+4x)(P^2_{n-2})].$$

Again investigating the phenomenon noted by Wingard, we proceed with the sequence D_n which is a C_4 joined by a single edge to an end vertex of a 2-cliquepath of length n.

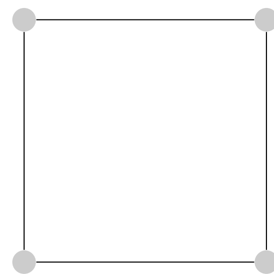

FIGURE 11. Z_0

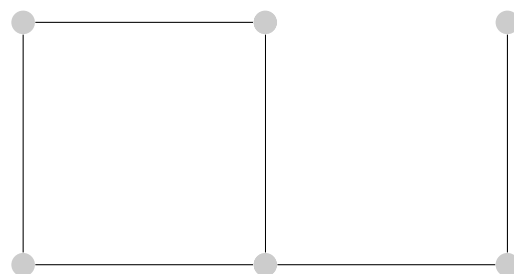

FIGURE 12. Z_1

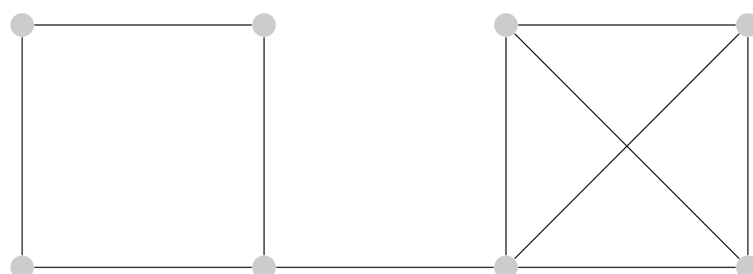

FIGURE 13. Z_2

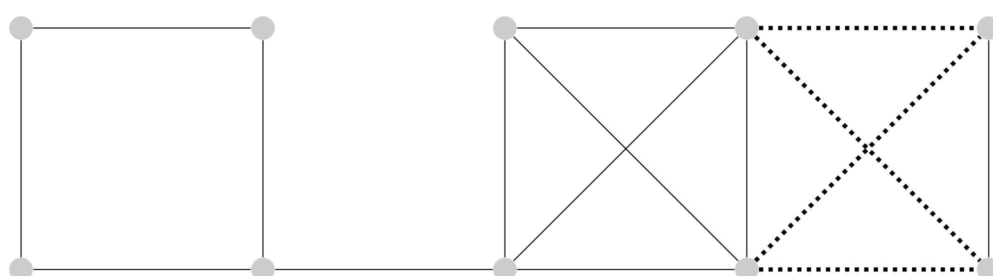

FIGURE 14. Z_n

$$Z_n(x) = Z_{n-1}(x) + 2xZ_{n-2}(x).$$

$$Z_0(x) = 1 + 4x + 2x^2.$$

$$Z_1(x) = 1 + 6x + 9x^2 + 3x^3.$$

$$\lambda^2 = \lambda + 2x.$$

$$\lambda_1 = \frac{1+\sqrt{1+8x}}{2}.$$

$$\lambda_2 = \frac{1-\sqrt{1+8x}}{2}.$$

$$Z_n(x) = f(x)\lambda_1^n + g(x)\lambda_2^n.$$

$$f(x) = \frac{1+4x+2x^2}{2} + \frac{1+8x+16x^2+6x^3}{2\sqrt{1+8x}}.$$

$$g(x) = \frac{1+4x+2x^2}{2} - \frac{1+8x+16x^2+6x^3}{2\sqrt{1+8x}}.$$

$$Z_n(x) = \frac{1}{2}[(1+4x+2x^2)(C_n^2(x)) + (1+8x+16x^2+6x^3)(P_{n-2}^2)].$$

Next we look at a 3-cliquepath. Several auxiliary classes of graphs will appear in the following derivation, these are shown in figures 17-19.

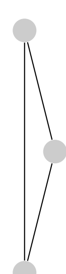

FIGURE 15. P_1^3

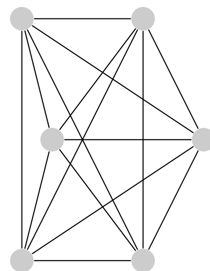

Figure 16. P_2^3

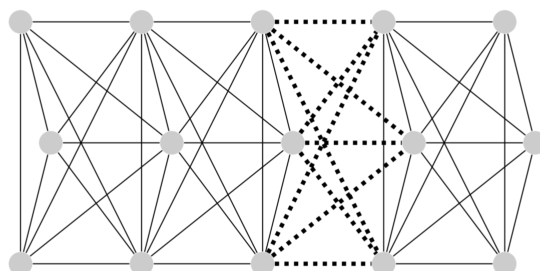

Figure 17. P_n^3

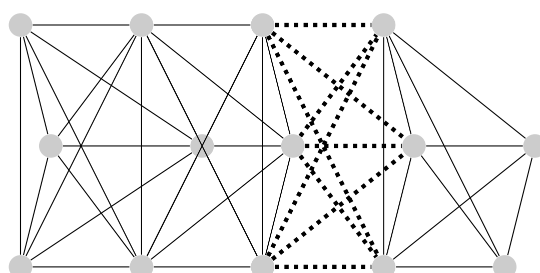

Figure 18. L_n

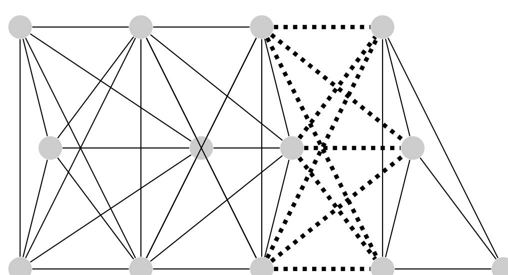

Figure 19. M_n

$$M_n(x) = P^3_{n-1}(x) + xP^3_{n-2}(x).$$

$$L_n(x) = M_n(x) + xL_{n-2}(x) = P^3_{n-1}(x) + 2xP^3_{n-2}(x).$$

$$P^3_n(x) = L_n(x) + xP^3_{n-2}(x) = P^3_{n-1}(x) + 3P^3_{n-2}(x).$$

$$P^3_0(x) = 1.$$

$$P^3_1(x) = 1 + 3x.$$

$$\lambda^2 = \lambda + 3x.$$

$$\lambda_1 = \frac{1+\sqrt{1+12x}}{2}.$$

$$\lambda_2 = \frac{1-\sqrt{1+12x}}{2}.$$

$$P^3_n(x) = f(x)\lambda_1^n + g(x)\lambda_2^n.$$

$$f(x) = \frac{1}{2} + \frac{1+6x}{2\sqrt{1+12x}}.$$

$$g(x) = \frac{1}{2} - \frac{1+6x}{2\sqrt{1+12x}}.$$

$$P^3_n(x) = \frac{1}{2^{n+1}}[(1)(1+\sqrt{1+12x})^n + (1-\sqrt{1+12x})^n) \\ + \frac{1+6x}{\sqrt{1+12x}}[(1+\sqrt{1+12x})^n - (1-\sqrt{1+12x})^n)]].$$

$$P^3_n(x) = \frac{1}{2}[(1)(C^3_n(x)) + (1+6x)(P^3_{n-2})].$$

Finally we get one more data point in looking at a path of 3-cliques attached to a 3 cycle.

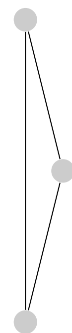

FIGURE 20. W_0

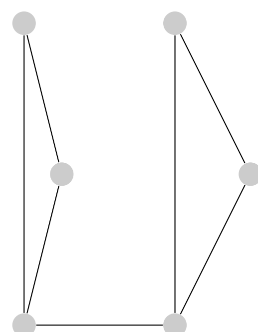

FIGURE 21. W_1

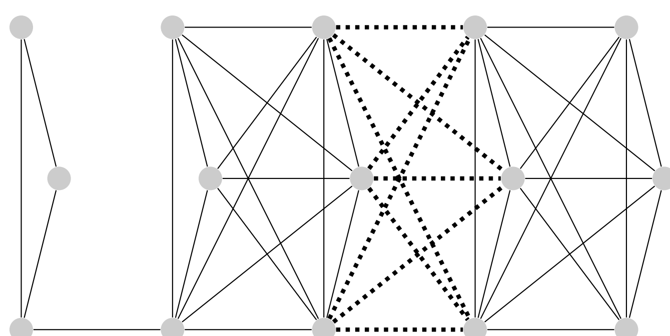

FIGURE 22. W_n

$$W_n(x) = W_{n-1}(x) + 3W_{n-2}(x).$$

$$W_0(x) = 1 + 3x.$$

$$W_1(x) = 1 + 6x + 8x^2.$$

$$\lambda^2 = \lambda + 3x.$$

$$\lambda_1 = \frac{1+\sqrt{1+12x}}{2}.$$

$$\lambda_2 = \frac{1-\sqrt{1+12x}}{2}.$$

$$f(x) = \frac{1+3x}{2} + \frac{1+9x+16x^2}{2\sqrt{1+12x}}.$$

$$g(x) = \frac{1+3x}{2} - \frac{1+9x+16x^2}{2\sqrt{1+12x}}.$$

$$W_n(x) = \frac{1}{2^{n+1}}[(1+3x)(1+\sqrt{1+12x})^n + (1-\sqrt{1+12x})^n)$$
$$+ \frac{1+9x+16x^2}{\sqrt{1+12x}}[(1+\sqrt{1+12x})^n - (1-\sqrt{1+12x})^n)]].$$

$$W_n(x) = \frac{1}{2}[(1+3x)(C_n^3(x)) + (1+9x+16x^2)(P_{n-2}^3)].$$

3. Extending the Reduction Formula

Now we turn our attention to the reduction formula itself. It is a powerful tool, but it has limitations, namely that it only removes one vertex at a time, requiring multiple steps to take out larger subgraphs. In developing a more robust reduction technique we introduce a few more concepts.

Definition 3.1. *Let $G(V,E)$ be a graph with $T \subset V$. T is said to be a clone set provided that for all $s,t \in T$, $N\{s\} = N\{t\}$. It is said to be a semi-clone set if s is adjacent to t and $N\{s\}\backslash t = N\{t\}\backslash s$. And it is said to be an external-clone set if $N\{s\}\backslash T = N\{t\}\backslash T$.*

Definition 3.2. *A path as defined before is a graph; a path IN a graph G, however, is a subgraph of G which is a path. A u,v-path is one such path whose endpoints are $u, v \in V(G)$.*

Definition 3.3. *A graph $G = (V,E)$ is connected if it has a u,v-path whenever $u, v \in V$. Otherwise G is disconnected.*

Definition 3.4. *The components of a graph G are its maximal connected subgraphs.*

Definition 3.5. *A separating set, or vertex cut, of a graph G is a set $S \subset V(G)$ such that $G \backslash S$ has more than one component. A min-cut S of G is a cut which can be no smaller in G, that is if less than $|S|$ vertices are deleted, G will remain connected.*

Theorem 3.6. *Independence polynomials are multiplicative on components, that is given a graph G with components A,B, $G(x) = A(x)B(x)$.*

Theorem 3.7. *(Basic independence polynomials)*
The independence polynomial of a single vertex is 1+x;
The independence polynomial of an independent set of size k is $(1+x)^k$;
And the independence polynomial of a k-clique is 1+kx.

Now logically we can break the independent sets in a graph, in relation to some subgraph S, into three categories: those sets which do not involve any element of S, those sets which involve at least one element of S but nothing in $N\{S\}$, and those sets, which may not exist, which involve at least one element of S and at least one element of $N\{S\}$. With this insight, we can write explicit formulas for the independence polynomials of certain classes of graphs.

Theorem 3.8. *Let G be a graph with a cut $S = \{v_1, v_2, \ldots, v_k\}$ that is a clone set. Let A,B be the components of $G \backslash S$. Then*

$$G(x) = A(x)B(x) + ((1+x)^k - 1)(A \backslash N\{S\})(x)(B \backslash N\{S\})(x).$$

Proof. Let $v \in S$.

To aid the reader in following the proof, the following conventions will be used:

$G' = G \backslash S$;

$H = G \backslash v$;

$L = H \backslash S, N\{S\} = G \backslash S, N\{S\}$.

Note that if S is a clone set, then S is also an independent set.

We proceed via induction on $|S|$. If $|S| = 1$ then by reduction on v we get that

$$G(x) = H(x) + xL(x) = H(x) + ((1 + x) - 1)L(x)$$

so the theorem holds. Suppose that it is true for $|S| = k - 1$ and consider the case of $|S| = k$. By reduction on v we get

$$G(x) = H(x) + x(1+x)^{k-1}L(x) = G'(x) + ((1+x)^{k-1} + x(1+x)^{k-1} - 1)L(x)$$

$$= G'(x) + ((1+x)(1+x)^{k-1} - 1)L(x)$$

□

Now a similar formula holds for a semi clone cut set.

Theorem 3.9. *Let G be a graph with a cut $S = \{v_1, v_2, \ldots, v_k\}$ which is a semi-clone set. Let A,B be the components of $G \backslash S$. Then*

$$G(x) = A(x)B(x) + ((1 + kx) - 1)(A \backslash N\{S\})(x)(B \backslash N\{S\})(x).$$

Proof. Note that if x, y are semiclones for all $x, y \in S$, then S is a k-clique.

Let $v \in S$

To aid the reader in following the proof, the following conventions will be used:

$G' = G \backslash S$;

$H = G \backslash v$;

$L = H \backslash S, N\{S\} = G \backslash S, N\{S\}$.

We proceed via induction on $|S|$. If $|S| = 1$ then by reduction on v we get that

$$G(x) = H(x) + xL(x) = H(x) + ((1 + x) - 1)L(x)$$

so the theorem holds. Suppose that it is true for $|S| = k - 1$ and consider the case of $|S| = k$. By reduction on v we get

$$\begin{aligned} G(x) = H(x) + xL(x) &= G'(x) + ((1 + (k-1)x) + x - 1)L(x) \\ &= G'(x) + ((1 + kx) - 1)L(x). \end{aligned}$$

□

These results can be generalized to the case of S being made up of external clones.

Theorem 3.10. *Let G be a graph with an external clone set S. Then we have*

$$G(x) = G\backslash_S(x) + (S(x) - 1)G\backslash_{S,N\{S\}}(x).$$

Proof. Let $v \in S$.

To aid the reader in following the proof, the following conventions will be used:

$G' = G \backslash S$;

$S' = S \backslash v$;

$H = G \backslash v$;

$L = H \backslash S, N\{S\} = G \backslash S, N\{S\}$;

$T = S \backslash v, N\{v\}$.

Note that S(x) = S' + xT(x).

We proceed via induction on $|S|$. If $|S| = 1$ then by reduction on v we get that

$$G(x) = H(x) + xL(x) = H(x) + ((1 + x) - 1)L(x)$$

so the theorem holds. Suppose that it is true for $|S| = k - 1$ and consider the case of $|S| = k$. By reduction on v we get

$$G(x) = H(x) + x(T(x))L(x) = G'(x) + (S'(x) - 1 + xT(x))L(x)$$
$$= G'(x) + (S(x) - 1)L(x).$$

□

4. A general formula

In this section a general expression will be derived for the independence polynomial of graphs formed by joining a given graph S_1 to a cliquepath. The joining is by a single edge between a vertex of S_1 and an end vertex of the cliquepath.

For convenience in the derivation, the following conventions will be used:

C_n^k is an n length k-cliquecycle;

P_n^k is an n length k-cliquepath;

S_2 is S_1 with a 2k-clique attached to a single vertex of S_1 via the vertices of the first k-clique;

$[S_1 : P_n^k]$ is S_1 with P_n^k attached by a single vertex of minimal connectivity to the vertex of S_1 referenced in S_2's definition;

$S_3 \equiv S_2$ with a k-clique rather than a 2k-clique;

$S_4 \equiv S_3$ minus an edge between S_1 and the clique;

$S_5 \equiv S_1$ minus the connected vertex from S_2;

$$P_n^k(x) = P_{n-1}^k + kxP_{n-2}^k;$$

$$C_n^k(x) = P_{n-1}^k + kxP_{n-3}^k;$$

$$S_2(x) = S_3(x) + kxS_1(x);$$

$$S_4(x) = S_1 + xS_5(k-1)xS_1;$$

$$S_3(x) = S_1 + 2xS_5 + (k-2)xS_1;$$

Example 2. The graph Ω_n^2, pictured for $n = 4$ in figure 27, consists of a 2-cliquepath attached to a 6-cycle. Several auxiliary graphs necessary in deriving the polynomial are shown in the surrounding figures.

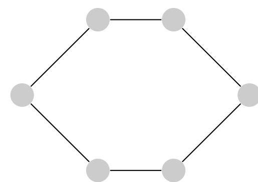

FIGURE 23. S_1

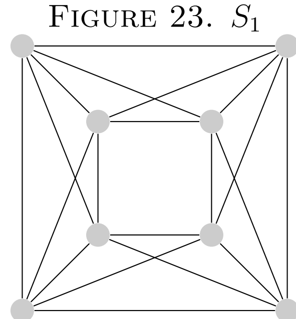

FIGURE 24. C_4^2

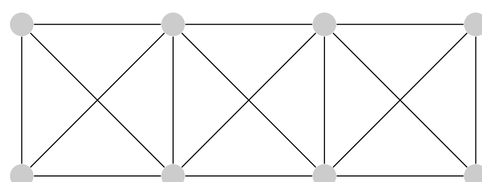

FIGURE 25. P_4^2

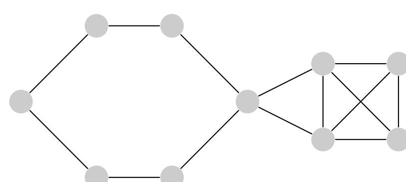

FIGURE 26. S_2

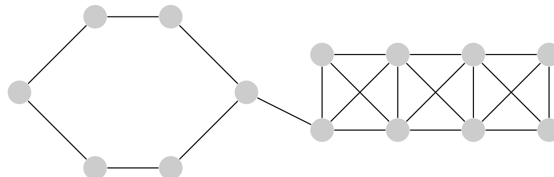

FIGURE 27. $[S_1 : P_4^2]$

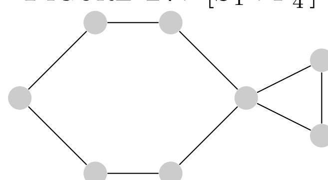

FIGURE 28. S_3

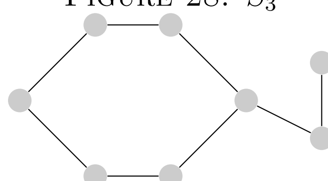

FIGURE 29. S_4

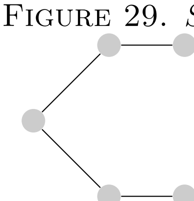

FIGURE 30. S_5

Theorem 4.1. *Let $[S_1 : P_n^k]$, P_n^k, C_n^k, S_2 be as above. Let S_1 be a graph. Then for $n \geq 2$*

$$[S_1 : P_n^k](x) = \frac{1}{2}[(S_1(x))(C_n^k(x)) + (S_2(x))(P_{n-2}^k(x))]$$

Proof. By using the generalized vertex reduction on the 2^{nd} k-clique in the chain we have

$$[S_1 : P_n^k](x) = S_4(x)(P_{n-2}^k(x)) + kx(S_1(x))(P_{n-3}^k).$$

Plugging into the equation we get

$$S_4(x)(P_{n-2}^k(x)) + kx(S_1(x))(P_{n-3}^k(x))$$

$$=1/2[(S_1(x))(P_{n-1}^k(x) + kxP_{n-3}^k(x)) + (S_3(x) + kxS_1(x))(P_{n-2}^k(x))]$$

$$\Longrightarrow 2S_4(x)(P_{n-2}^k(x)) + 2kx(S_1(x))(P_{n-3}^k(x))$$

$$=S_1(x)(P_{n-1}^k(x)) + S_1(x)(kxP_{n-3}^k(x)) + S_3(x)(P_{n-2}^k(x)) + kxS_1(x)(P_{n-2}^k(x))$$

$$\Longrightarrow 2S_4(x)P_{n-2}^k(x) + 2S_1(x)kxP_{n-3}^k(x)$$

$$=S_3(x)(P_{n-2}^k(x) + S_1(x)(P_{n-1}^k(x) + kxP_{n-2}^k(x) + kxP_{n-3}^k(x))$$

$$\Longrightarrow 2S_4(x)P_{n-2}^k = S_3(x)P_{n-2}^k(x) + S_1(x)(P_{n-1}^k(x) + kxP_{n-2}^k(x))$$

$$\Longrightarrow 2S_1(x)P_{n-2}^k(x) + 2xS_5(x)P_{n-2}^k(x) + 2kxS_1(x)P_{n-2}^k(x) - 2xS_1(x)P_{n-2}^k(x) + kxS_1(x)P_{n-3}^k(x)$$

$$=S_1(x)P_{n-2}^k(x) + 2xS_5(x)P_{n-2}^k(x) + kxS_1(x)P_{n-2}^k(x) - 2xS_1(x)P_{n-2}^k(x) + S_1(x)P_{n-1}^k(x) + kx$$

$$\Longrightarrow 2S_1(x)P_{n-2}^k(x) + kxS_1(x)P_{n-3}^k(x) = S_1(x)P_{n-2}^k(x)S_1(x)P_{n-1}^k(x)$$

$$\Longrightarrow 2S_1(x)P_{n-2}^k(x) + kxS_1(x)P_{n-3}^k(x) = 2S_1(x)P_{n-2}^k(x) + kxS_1(x)P_{n-3}^k(x).$$

□

5. Future Research

While the obtained generalization of the vertex reduction formula is quite useful, there remain many classes of graphs for which it is incomplete. Further investigation into this topic should likely revolve around attempting to find these missing pieces to extend it to more complex graphs.

6. Acknowledgments

Many thanks go out to those who helped with the completion of this thesis, whether by content or moral support. Specifically, I would like to thank the committee for their useful comments and critiques on my work. Moreover, my advisor, Dr. Staton, deserves special recognition for his immense help and encouragement from the very beginning of my research. Without his direction much of this would not have been possible.

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