THE TUTTE POLYNOMIAL FORMULA FOR THE CLASS OF TWISTED WHEEL GRAPHS by

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Abstract

The Tutte Polynomial Formula for the Class of Twisted Wheel Graphs

The 20th century work of William T. Tutte developed a graph polynomial that is modernly known as the Tutte polynomial. Graph polynomials, such as the Tutte polynomial, the chromatic polynomial, and the Jones polynomial, are at the heart of combinatorical and algebraic graph theory and can be used as a tool with which to study graph invariants. Graph invariants, such as order, degree, size, and connectivity which are each defined in Section 2, are graph properties preserved under all isomorphisms of a graph. Thus any graph polynomial is not dependent upon a particular labeling or drawing but presents relevant information about the abstract structure of the graph. The Tutte polynomial is the most general graph polynomial that satisifies the recurrence relationship of deletion and contraction. Deletion and contraction, collectively known as reduction operations and defined in Section 2, are two important actions that can be performed upon a graph in order to aide in the computation of the graph polynomial of interest. The deletion and contraction relationship states that for every edge e of a graph G, the polynomial of G equals the sum of the polynomial of G delete e and the polynomial of G contract e. Even with the help of these reduction operations, the Tutte polynomial of a graph can be hard to compute with only pen and paper, leading to occassions in which researchers approach the task of developing a formula for the Tutte polynomial of some family of graphs; i.e. a collection of graphs that adhere to common properties. In this thesis, we review the work necessary to compute the Tutte polynomial of the class of fan graphs and the class of wheel graphs and then add to this collection of known formulas by computing the formula for the Tutte polynomial of the class of twisted wheel graphs.

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1 Introduction

William T. Tutte was born during the summer of 1917 into modest beginnings in Newmarket, Suffolk, England, as the son of a gardener and a caretaker. As depicted by Arthur M. Hobbs and James G. Oxley in [9], Tutte was dedicated to his education at an early age and would travel upward of fifteen miles each morning by bike or by train to attend grade school and to satisfy his growing curiosity. Through the accomplishments of his passionate efforts, he received a scholarship to attend Trinity College, Cambridge, in the fall of 1935 where he began an undergraduate study in chemistry, although mathematics and his involvement in the Trinity Mathematical Society were his principal interests.

After a hiatus from the classroom at the mercy of World War II, Tutte published his 417-page thesis, "An algebraic theory of graphs," in 1948 through which he first introduced his dichromatic polynomial, now called the Tutte polynomial. This work traced his study of the chromatic polynomial and the flow polynomial, among other graph theory topics. Today, the Tutte polynomial is one of the more important applications of graph theory. It among other contributions carries the legacy of William T. Tutte throughout the history of graph and matroid theory [9].

The Tutte polynomial, which we will denote as $T_G(x, y)$, though $T(G; x, y)$ is also acceptable, has throughout graph theory history made great leaps since its 1948 introduction. Researchers have developed three equivalent definitions each complex in its own respect. The structure of the Tutte polynomial will be explained in detail in Section 3 but if a sneak preview is desired, please see Figure 5. This figure is an exercise in the most intuitive breakdown of a Tutte polynomial calculation.

As earlier noted, many researchers have taken on the task of generating recurrence relations for the Tutte polynomial of certain classes of graphs. We wish through this paper to add onto that list by developing a recurrence relation for the Tutte polynomial of the class of twisted wheel graphs. For a comprehensive look at classes whose Tutte polynomial formula is known, Criel Merino presents an extensive list in [11].

Along with satisfying the deletion and contraction recursive relationship, the

Tutte polynomial has known capabilities of counting graph invariants when x and y are specific values. For example, in [7], Joanna A. Ellis-Monaghan gives an explicit list of some of these counting capacities as the following,

Theorem 1 *If G is a connected graph then:*

1. $T_G(1,1)$ *equals the number of spanning trees of G,*
2. $T_G(2,1)$ *equals the number of spanning forests of G,*
3. $T_G(1,2)$ *equals the number of spanning connected subgraphs of G, and*
4. $T_G(2,2)$ *equals the number of subsets of edges of G.*

A longer list of these known evaluations can be found in Theorem 3.5 of [8]. In each of these invariant evaluations of the Tutte polynomial, discrete values, such as 1 and 2 in Theorem 1, are substituted for the x and y of $T_G(x,y)$ to determine the count of the considered invariant. These evaluations of the Tutte polynomial are the heart of its importance because they unlock graph properties through one polynomial that otherwise would have to be independently counted.

In this thesis, we will address fundamental definitions and preliminaries linked to the computation of the Tutte polynomial in Section 2. The next, Section 3, will introduce three equivalent interpretations of the Tutte polynomial and view trivial exercises of the definitions. Section 4 recreates the Tutte polynomial formulas of two classes of graphs that are the precursors for the formula for the Tutte polynomial of the class of twisted wheels, along with the development of the Tutte polynomial formula of our class of interest. Section 5 presents ideas for the further study of the class of twisted wheel graphs and the Tutte polynomial.

2 Graph Theory Preliminaries

Before discussion of the complex definitions and applications of the modern Tutte polynomial, it is important to do a thorough review of the applicable graph theory concepts and tools. Therefore, the next several pages will focus on the presentation of graph theory terminology such as degree, subgraph, and connection, all of which are fundamental to the theory of the Tutte polynomial.

A *graph* is an ordered pair of sets $G = (V, E)$ where E is a set of unordered two-element subsets of the finite set V [15]. The elements of V are called the *vertices* of the graph, and the elements of E are called the *edges* of the graph. Thus, each edge of E contain exactly two of the vertices of V. If G is a graph, then $V = V(G)$ is the *vertex set* of G, and similarly, $E = E(G)$ is the *edge set* of G. The *cardinality* of a set is the number of elements contained in that set. For the vertex set, the cardinality, denoted $|V(G)|$, is called the *order* of G. Dually, $|E(G)|$ is called the *size* of G. For vertices $x, y \in V$, the edge $\{x, y\}$ is said to be the *join* of x and y and can simply be written as $x\,y$ without the curly set braces [3]. Hence $x\,y$ and $y\,x$ are the exactly the same edge with vertices x and y as the *endpoints* [3].

$$G = \{\{a, b, c, d\}, \{\{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}\}$$

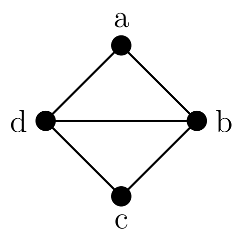


Figure 1: A graph.

A natural step in graph theory is to visualize a picture of the graph. For small graphs, the simplest way of comprehension is to draw it as in Figure 1. This graph is the collection of four vertices and five edges where any vertex that is a part of an edge is said to be *incident to* that edge. Since the edge $a\,b \in E(G)$, we say that a is incident to $a\,b$ and also that a is *adjacent* to, or *neighboring*, b. The notation $a \sim b$ means that a is adjacent to b. The set of all vertices adjacent to a is called the *neighborhood* of a and is denoted as $N(a)$. For clarification, $N(a) = \{b, d\}$ whereas $N(b) = \{a, c, d\}$ in Figure 1.

Building upon the concept of neighborhoods is the important tool of *degree.* For the graph $G = (V, E)$, the degree of a vertex $v \in V(G)$ is equal to the number of vertices in the neighborhood of v, i.e. the number of edges with which v is incident. This number is denoted as $d(v)$ or $d_G(v)$ if there is any risk of confusion. To connect these concepts to neighborhoods, we say that $d(v) = |N(v)|$. For the graph in Figure 1, we have

$$d(a) = 2 \quad d(b) = 3 \quad d(c) = 2 \quad d(d) = 3.$$

Since each edge of a graph has two endpoints, something special happens when we add together the degrees of a graph's vertices. That is,

$$\sum_{v \in V(G)} d(v) = 2|E(G)|. \tag{1}$$

An example of Equation (1) applied to Figure 1 is

$$\sum_{v \in V(G)} d(v) = d(a)+d(b)+d(c)+d(d) = 2+3+2+3 = 10 = 2\times 5 = 2|E(G)|. \tag{2}$$

Two important operations in Tutte polynomial calculations are *deletion* and *contraction.* Let G be a graph and $e \in E(G)$. To delete e is exactly as it sounds; that is, the edge is removed from the edge set with no effect on the vertex set. More rigorously, we can write $G - e = (V, E - e)$ to represent a graph after the deletion operation is performed. The contraction of e is the identification of the endpoints of e and the subsequent removal of e. The graph obtained after a contraction operation can be denoted as $G/e = (V/e, E - e)$. Both the edge set and the vertex set are altered in this case. Both the graphs $G - e$ and G/e are called *graph minors* of G, and the deletion and contraction operations are collectively known as *reduction operations.*

For more terminology on the description of specific types of graphs, we start with the graph on no edges commonly called the *edgeless graph* on n vertices denoted as E_n. An edgeless graph can be a single vertex, known as a *singleton*, or a collection of vertices with no incident edges. Thus each of these isolated vertices has a degree of 0. Another type of graph is the *multigraph*, which loosens the definition of the graph to allow *parallel edges* and *loops.* A parallel edge is a set of multiple edges which join the same two vertices; see the edges incident with the vertices c and d of Figure 2. For each parallel edge, a 1 is contributed to the degree of the incident vertex, i.e. $d(c) = 3$ for Figure 2 instead of $d(c) = 2$ as in the Figure 1 graph. A loop is defined as an edge joining a vertex to itself, as edge $b\,b$ in Figure 2. A loop edge contributes 2 to the degree of any vertex. Multigraphs are the natural context in which this text will focus as they are the natural state of our many of our graph's terminal forms.

$$G = \{\{a, b, c, d\}, \{\{a, b\}, \{a, d\}, \{b, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{d, c\}\}\}$$

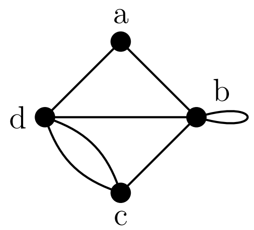
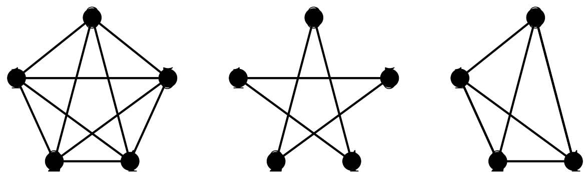


Figure 2: A multigraph.

We continue the discussion of graph theory preliminaries involved in Tutte polynomials with the *subgraph.* Informally, a subgraph is a graph contained in another graph, but when more strictly speaking, a graph H is a subgraph of graph G provided $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ where $\subseteq$ denotes subset. Hence, a subgraph H is formed by deleting various vertices and edges of graph G. If an edge $e \in E(G)$ is removed from graph G to form subgraph H, then $V(G) = V(H)$ because no vertices are affected. However, the same action results in $E(G) - \{e\} = E(H)$, a smaller edge set in size. This deletion is based solely upon an edge deletion, thus resulting in a special type of subgraph called a *spanning subgraph* because it includes or spans all of the vertices of $V(G)$. The only allowable deletions in the creation of a spanning subgraph are edge deletions; that is G and H must have the same vertex sets. A second type of subgraph is a the *induced subgraph.* Before giving its definition, we must recall that an edge is a two-element subset of vertices from $V(G)$. Therefore, it is obvious that if a vertex $v \in V(G)$ is deleted from graph G, then the edges in which v is an element must also be deleted. If not removed, then we you have edges that do not satisfy the two endpoint requirement of an edge. Now for the definition of induced subgraph. An induced subgraph is created solely by vertex deletion such that $V(H) \subseteq V(G)$ and $E(H) = \{xy \in E(H) | x \in V(H) \text{ and } y \in V(H)\}$ [3].



(a) G. (b) Spanning. (c) Induced.

Figure 3: Subgraphs.

Similar to the importance of subgraphs is the discussion of connectedness. In Theorem 1, we noted some counting capacities of the Tutte polynomial under the assumption that G is connected. So what does it mean for a graph to be connected? First, we define a *walk* in a graph G as a sequence of vertices $v_0, v_1, \ldots, v_n$ where each vertex, v_{i-1} neighbors the next vertex, v_i for $0 \leq i \leq n$. The walk is on $n+1$ vertices and is said to have a *length* of n [15]. A walk is said to be *closed* if $v_0 = v_n$; otherwise, it is said to be *open.* In relationship to the notation of the walk, a *path*, denoted P_n, is an open walk of length n upon distinct vertices; i.e. no vertex can be revisited and no edge can be used twice [15]. Thus the path having an initial vertex, v_0, and a terminal vertex, v_n, is called the $(v_0, v_n)-path$. These two definitions are important because they are the basis for what it means for one vertex to be *connected to* another vertex. For $v_0, v_n \in G$, we say that v_0 is connected to v_n if there is and $(v_0, v_n)-$path in G. Furthermore, a graph G is *connected* if for every pair of distinct vertices $\{x, y\} \in V(G)$ there is a path from x to y; otherwise the graph is *disconnected.*

We use the definitions of connected and disconnected to define what it means to be a *component* of a graph. A component is a maximal connected subgraph of G [3]. If a graph is edgeless, then each vertex is a component of the graph. Also, it is possible to have only one component of G, which implies that the graph is connected. The number of components of G is denoted as $\kappa(G)$. If the vertex deletion of $v \in V(G)$ increases the number of components of G, then v is a *cut vertex.* Similarly, $e \in E(G)$ is called a *cut edge* or *bridge* if $G - e$ is a disconnected graph with more components than G.

Now that we have discussed components, we can focus on *partitions* and the *multiplicative property* of the Tutte polynomial. One can partition a graph G into its components; that is, split the graph into pairwise disjoint maximal connected subgraphs whose union is G. Also, a graph G can be divided into maximal 2-connected subgraphs. A subgraph B of G is a *block* of G if it is either a bridge or a maximal 2-connected subgraph of G [3]. Any two blocks can have at most one vertex in common; thus if x and y are vertices in block B, then the deletion of $E(B)$ removes any (x, y)-path from graph G. If a $E(G)$ can be grouped into k blocks $B_1, B_2, \ldots, B_k$, then $E(B_1) \cap E(B_2) \cap \ldots \cap E(B_k) = \emptyset$, and in fact, $E(B_i) \cap E(B_j) = \emptyset$. This concept allows us to use the multiplicative property of the

Tutte polynomial that can greatly ease many hard Tutte polynomial calculations.

Proposition 1 *Let G be a graph and $E(G)$ be organized into k blocks with the edge sets $E(B_1), E(B_2), \ldots, E(B_k)$. Then it follows that the Tutte polynomial of G is the product of the Tutte polynomials of $B_1, B_2, \ldots, B_k$; that is,*

$$T_G(x, y) = T_{B_1}(x, y)T_{B_2}(x, y) \ldots T_{B_k}(x, y).$$

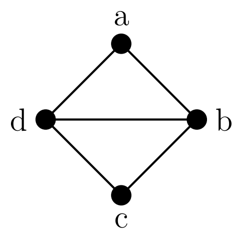
Proposition 1 can be found for example, in [3].

Similar to the walk and the path used in the discussion of subgraphs is the *cycle graph.* A cycle graph, denoted C_n, is a closed walk of length n across n vertices where the first and last vertex are the same but no other vertex is repeated. Conversely, we call a graph G with no cycles *acyclic.* An acyclic graph is more commonly known as a *forest.* A forest can be naturally described as a collection of *trees*, which are connected, acyclic graphs. Very important to Tutte's original definition of his polynomial is the *spanning tree.* A spanning tree of G can be defined as a spanning subgraph of G that is a tree. Note that in a tree, each edge is a cut edge [3]. It is known that every connected graph contains a spanning tree [15].

To build a relationship between all of these graph theory preliminaries and the Tutte polynomial, we begin by describing the *algebraic theory of graphs.* This branch of graph theory was the focus of Tutte's 1948 Trinity College thesis and varies from geometric, combinatoric, and algorithmic approaches to graph theory. Algebraic graph theory can be broken down into linear algebra, group theory, and graph invariants [3]. Our interests focus on the study of graph invariants. A *graph invariant* is a graph property that is preserved under all isomorphisms of a graph. That is, an invariant depends not on a particular drawing or labeling of a graph but upon the abstract structure of the graph. Many of the definitions we presented are examples of graph invariants including degree, order, size, and connectivity. *Graph polynomials*, such as the Tutte polynomial, are at the center of algebraic graph theory because they can be used as a tool with which to study graph invariants.

The *chromatic polynomial* is a one-variable graph polynomial that counts the number of ways vertices of a graph can be properly colored and is a good precusor to study of the Tutte polynomial. A *proper coloring* of the vertex set of a graph G is an assignment of colors to the vertices such that no two adjacent vertices receive the same color. That is, a map $c : V(G) \to \{1, 2, \ldots, x\}$ such that if vertices $u, v \in V(G)$ are adjacent, then $c(u) \neq c(v)$ [3]. The minimum number of colors required to properly color G is called the *chromatic number* and is denoted as $\chi(G)$. A famous graph theory question proposed in 1852 by Frances Gutherie asked how many colors it takes to color a map such that regions with a shared border receive different colors. A preliminary proof was first developed in 1972 by Kenneth Appel and Wolfgang Haken [8] and has matured into the following modern graph-theoretic terms that following are generally accepted as,

Theorem 2 *Four Color Theorem: Every planar graph can be properly colored using at most four colors.*



$$P(G;\lambda) = \lambda(\lambda-1)(\lambda-2)^2.$$

Figure 4: Chromatic polynomial.

The chromatic polynomial, denoted as $P(G;\lambda)$, counts the number of proper colorings of a graph G using λ or fewer colors. So $P(G;\lambda) > 0$ if and only if $\lambda \geq \chi(G)$. Thus $P(G;\lambda)$ is a polynomial in terms of λ. For an example of a chromatic polynomial calculation see Figure 4. In this example, there are λ choices for the color of vertex a. Then for vertex b, $\lambda - 1$ choices remain. For vertices c and d, $\lambda - 2$ color choices are available for their coloring. Hence the chromatic polynomial of the Figure 4 graph is $P(G;\lambda) = \lambda(\lambda-1)(\lambda-2)^2$.

The chromatic polynomial was initially related to the Tutte polynomial because it was a precursor of study to William T. Tutte's development of his dichromatic polynomial in 1940s. However, the chromatic polynomial can be obtained from the evaluation of the Tutte polynomial with $y = 0$. Similar to Definition 3.1.1 of the Tutte polynomial, the chromatic polynomial satisfies the deletion contraction recurrence relationship when G is a connected graph. In [3], we have the relationship between the two polynomials as

$$P(G;\lambda) = (-1)^{r(G)}\lambda^{\kappa(G)}T_G(1-\lambda, 0). \tag{3}$$

The proof for this relationship can be found in [3], and the notation $r(G)$ and $\kappa(G)$ will be explained in Section 3.2, the Rank-Nullity Generating Function. As ealier mentioned, the Tutte polynomial is the most general graph polynomial that satisfies the deletion and contraction recurrence realtionship. This is true because as the chromatic polynomial is a normalization of the Tutte polynomial at $y = 0$, other graph polynomials such as the Jones polynomial and the flow polynomial are also normalizations of the Tutte polynomial at a determined value.

Please consult reference [3] if more clarification of a graph theory preliminary to the Tutte polynomial is desired.

3 Definitions

A special characteristic of the two-variable Tutte polynomial is the depth of its equivalent definitions. These are of interest to the field of graph theory and other applied sciences because each has potential manipulations into other graph polynomials such as the chromatic polynomial as we showed in Equation (3). Beyond the chromatic polynomial, the Tutte polynomial can also be normalized to graph polynomials such as the flow polynomial, the reliability polynomial, and the Jones polynomial under certain assumptions. These polynomials help to make the Tutte polynomial interesting and relevant in discussions of colorings, knot theory, and statistical physics. In this section, we review three definitions and put them to practice by computing the Tutte polynomial of trivial graphs for clarification. We will continue to denote the Tutte polynomial as $T_G(x, y)$ and assume that our graphs are finite connected multigraphs with both loops and parallel edges allowed.

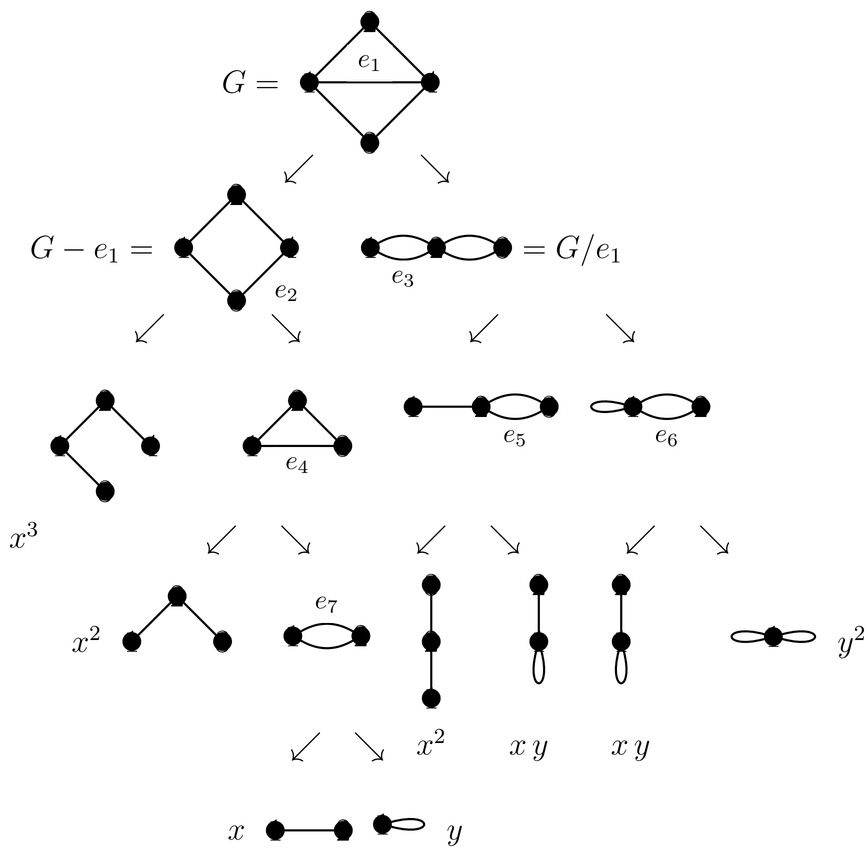
3.1 Deletion and Contraction

The first equivalent definition of the Tutte polynomial is by far the easiest to understand and is based upon the concept of the deletion and contraction recurrence relationship, $n(G) = n(G - e) + n(G/e)$ for a graph G. Before presenting the definition of the deletion and contraction method, this terminology is clarified. Recall, the graph obtained by the deletion of an $e \in E(G)$ of a graph G is $G - e = (V, E - e)$, and the graph obtained by the contraction of e is denoted as $G/e = (V/e, E - e)$. These basic definitions are used in the linear recurrence relation to rewrite a graph in smaller or simpler forms, which are graph minors. Then, by applying the same reduction rules to the newly generated graph minors, the method proceeds until reaching the most simple terminal forms. These terminal graphs are a series of forests with loops each identified with a monomial of independent variables and then summed to yield the complete graph polynomial [4]. Hence we can interpret the notation $n(G) = n(G - e) + n(G/e)$ as a representation of the graph polynomial, $n(G)$, that is equal to the sum of the polynomial of the graph minor after deletion of the edge e, $n(G-e)$, and the polynomial of the graph minor after the contraction of the edge e, $n(G/e)$. It is important to note that the resulting graph polynomial is independent of the order in which edges are chosen to undergo the reduction operations.

Definition 3.1.1 *If $G = (V, E)$ and e is an element of $E(G)$, then*

$$T_G(x,y) = \begin{cases} 1 & \text{if } E(G) = \emptyset \\ x\,T_{G-e}(x,y) & \text{if } e \in E(G) \text{ and } e \text{ is a bridge} \\ y\,T_{G/e}(x,y) & \text{if } e \in E(G) \text{ and } e \text{ is a loop} \\ T_{G-e}(x,y) + T_{G/e}(x,y) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

Figure 5 is an example of the Deletion and Contraction definition where the edges denoted $e_1, e_2, \ldots e_7$ represent the edges chosen to undergo each subsequent reduction operation before a terminal graph minor is reached.



$$T_G(x,y) = x^3 + 2x^2 + x + 2xy + y + y^2$$

Figure 5: An example of computing the Tutte polynomial of a graph recursively by using deletion and contraction.

This deletion and contraction method of computation is in many cases the most intuitive version of the Tutte polynomial definition. Throughout Section 4, this definition is applied as opposed to the Spanning Tree Expansion and the Rank-Nullity Generating Function definitions. Deletion and contraction operations naturally lead to linear recurrence relations and thus are the most helpful when solving for the formula for the Tutte polynomial of a class of graphs.

As a first example of deletion and contraction used to solve the formula for the Tutte polynomial of a family of graphs, we consider the class of cycle graphs.

3.1.1 Cycle Graphs

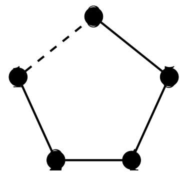


Figure 6: C_n.

We previously defined a cycle graph, C_n, as a closed walk on n distinct vertices. When you delete an edge e from $E(C_n)$, the resulting graph minor is a path on n vertices, denoted P_n, whose Tutte polynomial is x^{n-1}. While contracting an edge of C_n yields a smaller cycle on $n-1$ vertices, so $T(C_n) = T(P_n) + T(C_{n-1})$. This process of deletion and contraction can be continued until reaching the 1-cycle graph, C_1, that is simply and loop whose Tutte polynomial is known to be y. Hence the Tutte polynomial of C_n is the sum of $T(P_n) + T(P_{n-1}) + \ldots + T(P_2) + y$. A more concise formula can be found in [11] as follows,

$$T_{C_n}(x, y) = \sum_{i=1}^{n-1} x^i + y. \tag{4}$$

3.2 Spanning Tree Expansion

This second definition for the Tutte polynomial reflects W. T. Tutte's original work on his polynomial. Before presenting the concept, we introduce some relevant terminology.

The Spanning Tree Expansion definition, taken from [3], leads to an expansion in terms of spanning trees of the graph G. Then for a spanning tree S of G and an edge $e_j \in \big(E(G) - E(S)\big)$, there is a cycle defined by $E(S) \cup e_j$. We define $Z_{S(e_j)}$ as the set of edges in $E(S)$ whose union with e_j creates this unique cycle. We can more rigorously write

$$Z_{S(e_j)} = \{e_i \in E(S) : (S - e_i) + e_j \text{ is a spanning tree}\}.$$

Similarly, for an edge $e_i \in E(S)$, there is a cut defined by $\big(E(G) - E(S)\big) \cup e_i$ such that if e_i and $E(G) - E(S)$ are removed from G then G is no longer a connected graph. We write $U_{S(e_i)}$ as the set of edges of $E(G) - E(S)$ whose union with e_i creates this unique cut. That is,

$$U_{S(e_i)} = \big\{e_j \in \big(E(G) - E(S)\big) : (G - e_i) - e_j \text{ increases } \kappa(G)\big\}.$$

Let us now assume a fixed ordering $\prec$ on the edges of G, say $E(G) = \{e_1, e_2, \ldots, e_x\}$ where $e_i < e_j$ if and only if $i < j$. Call an edge $e_i \in E(S)$ an *internally active* edge of S with respect to the ordering of G if e_i is the smallest edge of the cut it defines, that is, e_i is internally active if $i \leq j$ whenever $e_j \in U_{S(e_i)}$. Dually, an edge $e_j \in E(G) - E(S)$ is an *externally active* edge if e_j is the smallest edge of the cycle it defines, that is, $i \geq j$ whenever $e_i \in Z_{S(e_j)}$. The *internal activity* of S, denoted as n, is the number of internally active edges, and the *external activity* is the number of externally active edges, denoted m. We call a spanning tree of the internal activity of n and the external activity of m an (n,m)*-tree* [3].

Definition 3.2.1 *If $G = (V, E)$ with a total ordering on its edge set and $e_x \in E(G)$, then*

$$T_G(x,y) = \begin{cases} 1 & \text{if } E(G) = \emptyset \\ x \sum_{n,m} t'_{n,m} x^n y^m & \text{if } e_x \in E(S) \text{ and } e_x \text{ is a bridge} \\ y \sum_{n,m} t'_{n,m} x^n y^m & \text{if } e_x \in E(G) - E(S) \text{ and } e_x \text{ is a loop} \\ \sum_{n,m} t'_{n,m} x^n y^m + \sum_{n,m} t''_{n,m} x^n y^m & \text{if } e_x \text{ is neither a bridge nor a loop} \end{cases}$$

where $t_{n,m}$ is the number of (n,m)-trees, $t'_{n,m}$ is the number of (n,m)-trees in $G - e_x$, and $t''_{n,m}$ is the number of (n,m)-trees in G/e_x.

It is important to note that just as the Deletion and Contraction definition is independent of reduction operation order, the coefficients of $t_{n,m}$ are independent of the total ordering and the choice of e_x. An application of the Spanning

Tree Expansion definition of the Tutte polynomial is presented using the Tutte polynomial of a member of the class of complete graphs.

3.2.1 Complete Graphs

A *complete graph* is a graph in which every pair of distinct vertices is connected by a unique edge. We will denoted these graphs as K_n where n is the order of the graph. Figure 7 depicts three complete graphs with $4 \leq n \leq 6$, respectively. The K_4 member of this class offers good practice in the spanning tree expansion definitions of the Tutte polynomial because its recreation is trivial.

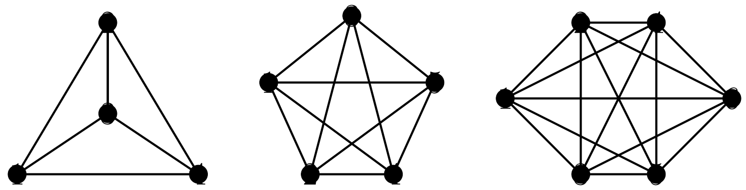


Figure 7: K_n.

We begin this exercise by labeling a fixed ordering of the edges of the graph as $e_1, e_2, \ldots, e_{\frac{n(n-1)}{2}} = e_1, e_2, \ldots, e_6$. In building a spanning tree of any complete graph on n vertices, we know that $n-1$ edges are necessary. Thus to count all the subgraphs of size $n-1$ for any K_n, we know that we will have $\binom{\frac{n(n-1)}{2}}{n-1}$ possibilities. For the K_4 graph, there are $\binom{6}{3} = 20$ possible subgraphs. However, for K_4, 4 of the possible subgraphs will be cycles of order 3 that will do not satisfy the definition of a spanning tree. Thus, there are $20 - 4 = 16$ possible spanning trees of K_4. In making a list of all 20 subgraph subsets, I chose to use a generating algorithm of lexicographic order. However, any method of generating the $(n-1)$-combinations of $\left\{e_1, e_2, \ldots, e_{\frac{n(n-1)}{2}}\right\}$ is permitted. Again, it is important then to throw out any $(n-1)$-combinations that create a cycle that does not span the graph, such as $\{e_1, e_4, e_5\}$ in Figure 8.

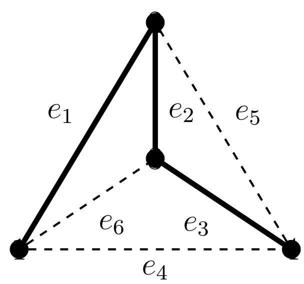


Figure 8: Representation of the Spanning Tree Expansion definition on K_4.

In the figure above, we have an example of a spanning tree of K_4 in which we have chosen $E(S) = \{e_1, e_2, e_3\}$ and $E(G) - E(S) = \{e_4, e_5, e_6\}$. It is obvious that for this spanning subgraph that e_1, e_2, and e_3 are internally active edges and that there are no externally active edges. Thus $n = 3$ and $m = 0$. Hence this subgraph of K_4 is a $(3, 0)$-forest and is counted in $t_{3,0} = 1$. Continue with the next subgraph in which $E(S) = \{e_1, e_2, e_4\}$ and $E(G) - E(S) = \{e_3, e_5, e_6\}$. For this spanning tree, e_1 and e_2 are internally active and there are no externally active edges. Thus, this is a $(2, 0)$-forest with $t_{2,0}$. Calculation of the internal and external edges of each subgraph is continued for the remaining 14 spanning trees of K_4. In total, K_4 has one $(3, 0)$-forest, three $(2, 0)$-forests, two $(1, 0)$-forests, four $(1, 1)$-forests, two $(0, 1)$-forests, three $(0, 2)$-forests, and one $(0, 3)$-forest; i.e. $t_{3,0} = 1$, $t_{2,0} = 3$, $t_{1,0} = 2$, $t_{1,1} = 4$, $t_{0,1} = 2$, $t_{0,2} = 3$, and $t_{0,3} = 1$. Therefore,

$$
\begin{aligned}
T_{K_4}(x, y) &= \sum_{n,m} t_{n,m} x^n y^m \\
&= 1\,x^3y^0 + 3\,x^2y^0 + 2\,x^1y^0 + 4\,x^1y^0 + 2\,x^0y^1 + 3\,x^0y^2 + 1\,x^0y^3 \\
&= x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3.
\end{aligned}
\tag{5}
$$

For a formula for the entire class of complete graphs, please see [11].

3.3 The Rank-Nullity Generating Function

The last and least intuitive definition for the Tutte polynomial is a generating function based on the notions of *rank* and *nullity*. A generating function is a polynomial whose coefficients count structures that are embedded in the polynomial's exponents [7]. This counting capacity makes this definition important in the field of combinatorics and to the known graph invariant applications of the Tutte polynomial such as those of Theorem 1. Before presenting the Rank-Nullity Generating Function definition taken from [7], some notation is clarified. Given a graph $G = (V, E)$ with the number of components denoted as $\kappa(G)$, the *cycle rank* of G can be expressed as $r(G) = |V(G)| - \kappa(G)$. Similarly, the *cycle rank* of F is $r(F) = |V(F)| - \kappa(F)$ where F is a subset of edges of graph G that induce a spanning subgraph of G. Lastly, we write the *nullity* of F as $n(F) = |E(F)| - r(F)$. This terminology is used in establishing the Rank-Nullity Generating Function definition as follows,

Definition 3.3.1 *If $G = (V, E)$ and $F \subseteq E(G)$, then*

$$T_G(x, y) = \sum_{F \subseteq E(G)} (x-1)^{r(G)-r(F)}(y-1)^{n(F)}.$$

Two special cases of this generating function definition are important to note. First, the *rank 0 graph* is the singleton graph with one loop edge. Second, the *rank 1 graph* is a graph of two vertices connected by a single edge [7].

An example of Definition 3.3.1 upon the K_4 graph is presented for comparison to the previous Spanning Tree Expansion definition application. This example will help to confirm the equivalence of the definitions and to show the complexity of Tutte polynomial calculations.

3.3.1 Complete Graphs for Comparison

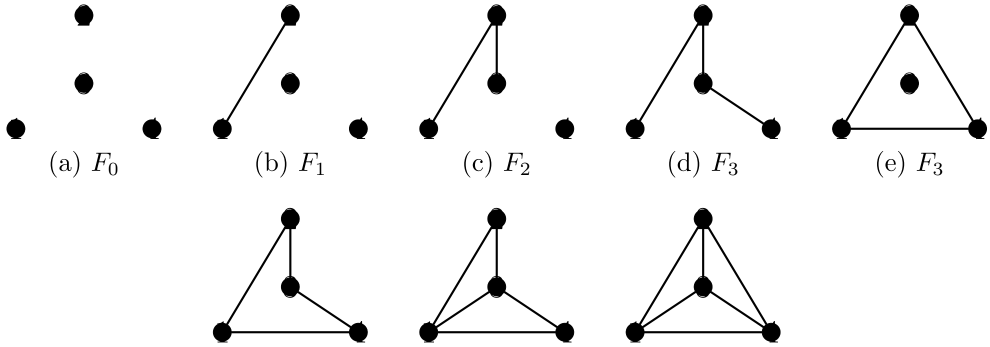
In applying the Rank-Nullity Generating Function definition, we begin by computing the rank and nullity of all of the possible spanning subgraphs of the graph of interest. Recall, that a spanning subgraph F of G must have the same order as G. Thus, in the computation of each subgraph F of G using the Rank-Nullity Generating Function Definition, we define each subgraph by the number of edges chosen. For the K_4 graph, possible spanning subgraphs $F_0, F_2, \ldots, F_i$ with $0 \leq i \leq 6$ are partitioned into disjoint sets based on size and $\kappa(F_i)$. To count the spanning subgraphs of size 0 or the subgraphs with no edges, calculate $\binom{6}{0}$. Hence there is only one spanning subgraph with 0 edges. To compute the number of subgraphs of size 1, calculate $\binom{6}{1}$. This calculation tells that there are six subgraphs of G with one edge. The number of spanning subgraphs for the remaining sizes can be calculated in a similar fashion as fifteen subgraphs of size 2, twenty subgraphs of size 3, fifteen subgraphs of size 4, six subgraphs of size 5, and one subgraph of size 6. Each F_i partition contains subgraphs with an equal number of components except partition F_3. These twenty sets must be further partitioned into those with $\kappa(F_3) = 1$ and $\kappa(F_3) = 2$; i.e. there are sixteen and four sets in each partition, respectively.

With these partitions established, the rank and nullity of each set of subgraphs can be computed as follows,

- The F_0 subgraph has $r(F_0) = 0, n(F_0) = 0$, and $\kappa(F_0) = 4$;
- The F_1 subgraphs have $r(F_1) = 1, n(F_1) = 0$, and $\kappa(F_1) = 3$;
- The F_2 subgraphs have $r(F_2) = 2, n(F_2) = 0$, and $\kappa(F_2) = 2$;
- The F_3 subgraphs with $\kappa(F_3) = 1$ have $r(F_3) = 3$ and $n(F_3) = 0$;
- The F_3 subgraphs with $\kappa(F_3) = 2$ have $r(F_3) = 2$ and $n(F_3) = 1$;

- The F_4 subgraphs have $r(F_4) = 3, n(F_4) = 1$, and $\kappa(F_4) = 1$;
- The F_5 subgraphs have $r(F_5) = 3, n(F_5) = 2$, and $\kappa(F_5) = 1$;
- The F_6 subgraphs have $r(F_6) = 3, n(F_6) = 3$, and $\kappa(F_6) = 1$.

In Figure 9, we show a representative of each partition listed above. Note that F_i for $i \in \{1, 2, \ldots, 6\}$ is a set of graphs, not a single graph.



(a) F_0 (b) F_1 (c) F_2 (d) F_3 (e) F_3

(f) F_4 (g) F_5 (h) F_6

Figure 9: Representation of the Rank-Nullity Generating Function definition of K_4.

These details allow Definition 3.3.1 to be exercised as follows,

$$\begin{aligned} T_{K_4}(x, y) &= \sum_{F \subseteq E(G)} (x-1)^{r(G)-r(F)} (y-1)^{n(F)} \\ &= 1\,(x-1)^{3-0}(y-1)^0 + 6\,(x-1)^{3-1}(y-1)^0 + 15\,(x-1)^{3-2}(y-1)^0 \\ &\quad + 16\,(x-1)^{3-3}(y-1)^0 + 4\,(x-1)^{3-2}(y-1)^1 + 15\,(x-1)^{3-3}(y-1)^1 \\ &\quad + 6\,(x-1)^{3-3}(y-1)^2 + 1\,(x-1)^{3-3}(y-1)^3 \\ &= x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3. \end{aligned} \tag{6}$$

This calculation shows that the Tutte polynomial for K_4 when applying the Rank-Nullity Generating Function definition is exactly that of the Tutte polynomial calculated when applying the Spanning Tree Expansion definition further confirming their equivalence. Also, this calculation gives a little hint as to how complex these calculations can become as the graph of interest grows and therefore the relevance of formulas for calculating the Tutte polynomial of classes of graphs.

4 The Tutte Polynomial of the Twisted Wheel

In order to complete an evaluation of the Tutte polynomial for a complex family of graphs, one must be equipped with the necessary tools. In many cases, this means prior calculation of the Tutte polynomial formulas of related families of graphs. Sometimes, the required family may be detailed in a compilation such as [11], otherwise evaluations can be a tricky and time-consuming task. For our purposes, two families of graphs are needed to complete the formula of the Tutte polynomial of the twisted wheel graphs. These two families are the class of fan graphs and the class of wheel graphs. And luckily for us, both of these families are recursive sequences whose Tutte polynomials satisfy a linear homogenous recursion relation and have been previously computed, which makes their calculation easier than most. Before presenting the Tutte polynomial formula for twisted wheel graphs, we recreate the evaluation techniques and recurrence relations of the Tutte polynomial of the classes of fan graphs and wheel graphs to provide a better understanding of their relationship.

4.1 Fan Graphs

A *fan graph*, denoted F_n, is defined as the graph join between E_m, the edgeless graph on m vertices, and P_n, the path on n vertices. For our purposes, m will always be equal to 1, thus $E_m = E_1$ is a singleton graph, which we will refer to as the *hub*. The n edges that join the hub to P_n are known as *spokes*, and the number of vertices of P_n or the number of spokes distinguishes between the members of the fan graph family. The dashed representation of two spokes and a rim edge in Figure 10 depicts the arbitrary size of F_n, which is not necessarily F_5 as the graph may appear. The illustration simply means that there is the possibility of more rim vertices and thus joining spokes and rim edges than can be drawn with pen and paper. This convention is carried throughout the remainder of this paper in application to the classes of graphs that have yet to be discussed.

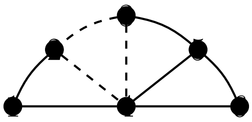
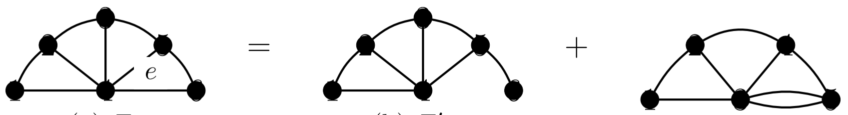


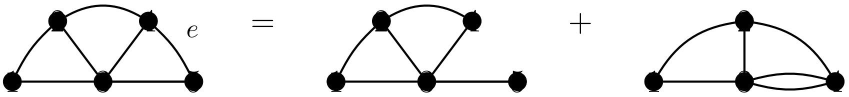
Figure 10: F_n.

Using the known fact that any connected graph G follows a recurrence model of $n(G) = n(G-e) + n(G/e)$, it is possible to claim two basic relationships on the sequence of fan graphs. These relations, as seen in Figure 11, are used as algebraic tools to solve for the Tutte polynomial formula for the class of fan graphs. We use the notation F'_n for F_n with an additional pendant edge and the notation F_n is used to denote that F^+_n has a parallel spoke. Also, we will see later that $\widetilde{F}_n$ denotes that the graph F_n has an edge that is a loop. It is important to note that the following graphical representations in Figure 11 are the shorthand notation for the Tutte polynomial of the corresponding graph. This convention is used for solving both the fan graph and later the wheel graph Tutte polynomial linear recurrence.



(a) F_n (b) F'_{n-1} (c) F^+_{n-1}

That is, $T_{F_n}(x,y) = T_{F'_{n-1}}(x,y) + T_{F^+_{n-1}}(x,y)$.

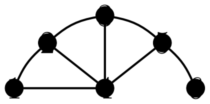


(d) F_{n-1} (e) F'_{n-2} (f) F^+_{n-2}

That is, $T_{F_{n-1}}(x,y) = T_{F'_{n-2}}(x,y) + T_{F^+_{n-2}}(x,y)$.

Figure 11: Basic Relationships used in the recurrence relation for the Tutte polynomial of the fan graph.

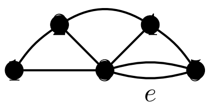
Now, we begin to build the linear recurrence by considering the first of the two basic relationships. Then by individually considering F'_{n-1} and F^+_{n-1}, the details of the fan graph Tutte polynomial formula unfold as shown in the following Figure 12 and Figure 13, respectively. We use a method of algebraic substitution taken from [10]. This method is most helpful in its application of the shorthand notation used to represent the Tutte polynomial of the depicted graph and will be used throughout future sections.



(a) F'_{n-1}

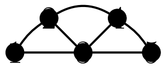
$$= \quad x\,T_{F_{n-1}}(x,y) \tag{7}$$

Figure 12: Tutte polynomial recurrence relation for graph $F_n - e$, i.e. F'_{n-1}.



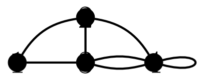
(a) F^{+}_{n-1}

$=$



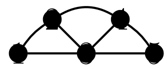
(b) F_{n-1}

$+$



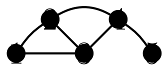
(c) $\widetilde{F}^{+}_{n-2}$

$= T_{F_{n-1}}(x,y) \quad + \quad y\Bigg($



(d) F_{n-1}

$-$



(e) F'_{n-2}

$\Bigg)$

$$\begin{aligned} &= T_{F_{n-1}}(x,y) + y\,T_{F_{n-1}}(x,y) - x\,y\,T_{F_{n-2}}(x,y) \\ &= (1+y)\,T_{F_{n-1}}(x,y) - x\,y\,T_{F_{n-2}}(x,y) \end{aligned} \tag{8}$$

Figure 13: Tutte polynomial recurrence relation for the graph F_n/e, i.e. F^{+}_{n-1}.

Thus the recurrence relation for the Tutte polynomlial formula for the graphs of F_n is solved by the summation of Equation (7) and (8) as

$$T_{F_n}(x,y) = (x+y+1)\,T_{F_{n-1}}(x,y) - x\,y\,T_{F_{n-2}}(x,y). \tag{9}$$

This is a linear homogeneous recurrence relation of order 2 with constant coefficient from which we get a characteristic polynomial,

$$\lambda^2 - (x+y+1)\,\lambda + x\,y = 0, \tag{10}$$

for which two roots can easily be solved as λ_1 and λ_2 are

$$\frac{x+y+1 \pm r}{2}, \quad \text{when } r = \sqrt{(x+y+1)^2 - 4xy}.$$

Thus, by using the Tutte polynomials of F_1 and F_2 given in Figure 14 as the initial conditions, coefficients can be determined that satisfy the linear recurrence and a particular solution can be obtained as

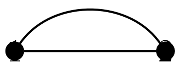
$$
\begin{aligned}
T_{F_n}(x,y) = &\left(\frac{1+x^2+y-r+x\,(-y+r)}{2\,x\,r}\right)\left(\frac{x+y+1+r}{2}\right)^n \\
&+\left(-\frac{1+x^2+y+r-x\,(y+r)}{2\,x\,r}\right)\left(\frac{x+y+1-r}{2}\right)^n
\end{aligned}
\tag{11}
$$

for all $n \geq 1$.

An example of Equation (11) on the fan graph of 4 spokes would be as follows,

$$
\begin{aligned}
T_{F_4}(x,y) = &\left(\frac{1+x^2+y-r+x\,(-y+r)}{2\,x\,r}\right)\left(\frac{x+y+1+r}{2}\right)^4 \\
&+\left(-\frac{1+x^2+y+r-x\,(y+r)}{2\,x\,r}\right)\left(\frac{x+y+1-r}{2}\right)^4 \\
= &\ x+3\,x^2+3\,x^3+x^4+y+4\,x\,y+3\,x^2\,y+2\,y^2+2\,x\,y^2+y^3.
\end{aligned}
\tag{12}
$$

We have not been able to find a scholarly article that solves for the Tutte polynomial of the class of fan graphs using the Deletion and Contraction definition in this manner. However in [5], Brennen approaches this class of graph with Definition 3.3.1 applying a generating function to compute the Tutte polynomial formula for this class. Both Equation (11) and her presented formula, though they look very different, give equivalent results and further illustrate the equivalence of the three Tutte polynomial defintions discussed in Section 3.

$T_{F_1}(x,y)$ =  = $x+y$

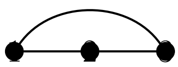
$T_{F_2}(x,y)$ =  = x^2+x+y

Figure 14: First two Tutte polynomial terms from the sequence $\{F_n\}$.

4.2 Wheel Graphs

A *wheel graph*, W_n, is a graph that contains a cycle graph of order n for which every vertex of C_n is adjacent to an additional singleton, E_1. Similar to the fan graph, the singleton is known as the hub and the edges incident to the hub are known as spokes. The W_n graph is illustrated in Figure 15.

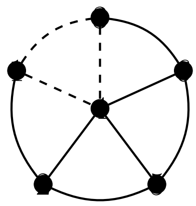
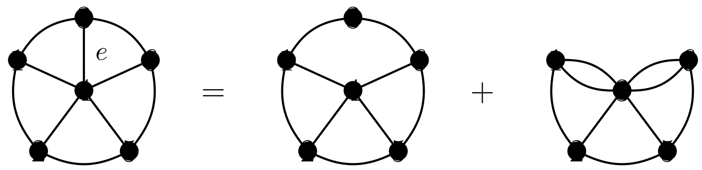
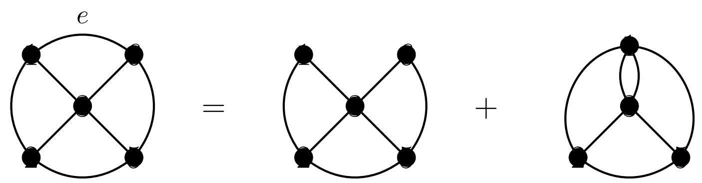


Figure 15: W_n.



(a) W_n (b) F^{-}_{n-1} (c) F^{++}_{n-1}

That is, $T_{W_n}(x,y) = T_{W_n^-}(x,y) + T_{F^{++}_{n-1}}(x,y)$.

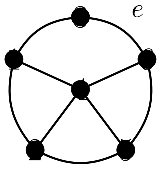


(d) W_{n-1} (e) F_{n-1} (f) W^{+}_{n-2}

That is, $T_{W_{n-1}}(x,y) = T_{F_{n-1}}(x,y) + T_{\widetilde{W}_{n-2}}(x,y)$.

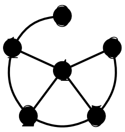
Figure 16: Basic relationships used in the recurrence relations for the Tutte polynomial of the wheel graph.

Following the construction pattern of the class of fan graphs, the formula for the Tutte polynomial of wheel graphs can be computed using the Deletion and Contraction definition and algebraic manipulations on a set of basic relationships. These relationships are seen in Figure 16 above. In these graphs, we will use F_n^- to denote F_n in which the two end rim vertices are both adjacent to a single vertex that is not a part of F_n, see Figure 16b. Similar to the recurrence notation of the fan graphs, we denote a wheel graph in which W_n has a parallel spoke as W_n^+ and thus the notation F_n^{++} is used to denote that F_n has two sets of parallel spokes, i.e. Figure 16c. Also, F_n' is still used to denote that F_n has a pendant edge. Instead of order two linear homogenous recurrence relation, a linear homogeneous recurrence relation of order three with constant coefficients is implemented for the class of wheel graphs. We begin building this recurrence based upon the first relation of Figure 16. Now, seperately consider the graphs F^{-}_{n-1} and F^{++}_{n-1} as follows



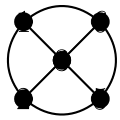
(a) F^{-}_{n-1}

$=$



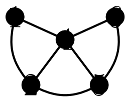
(b) F'_{n-1}

$+$



(c) W_{n-1}

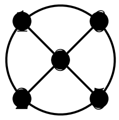
$= \; x \Big($



$\Big) + T_{W_{n-1}}(x,y)$

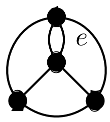
(d) F_{n-1}

$= \; x \Big($



(e) W_{n-1}

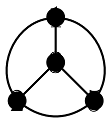
$-$



(f) W^{+}_{n-2}

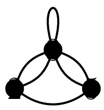
$\Big) + T_{W_{n-1}}(x,y)$

$= \; x\, T_{W_{n-1}}(x,y) - x \Big($



(g) W_{n-2}

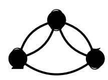
$+$



(h) $\widetilde{F}^{++}_{n-3}$

$\Big) + T_{W_{n-1}}(x,y)$

$= \; (x+1)\, T_{W_{n-1}}(x,y) - x\, T_{W_{n-2}}(x,y) - x\, y \Big($

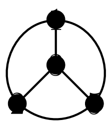


(i) F^{++}_{n-3}

$\Big)$

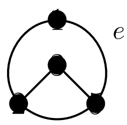
$= \; (x+1)\, T_{W_{n-1}}(x,y) - x\, T_{W_{n-2}}(x,y)$

$-x\, y \Big($



(j) W_{n-2}

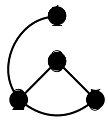
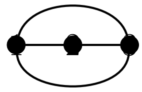
$-$



(k) F^{-}_{n-3}

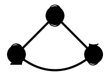
$\Big)$

$$= \quad (x+1)\, T_{W_{n-1}}(x,y) - (x\, + x\, y)\, T_{W_{n-2}}(x,y)$$

$$+x\, y \Big($$  $+$  $\Big)$

(l) F'_{n-3} (m) W_{n-3}

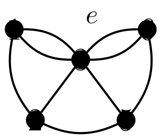
$$= \quad (x+1)\, T_{W_{n-1}}(x,y) - (x\, + x\, y)\, T_{W_{n-2}}(x,y) + x\, y\, T_{W_{n-3}}(x,y)$$

$$+x^2\, y \Big($$  $\Big)$

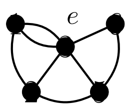
(n) F_{n-3}

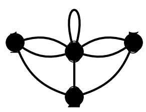
(13)

Figure 17: Tutte polynomial recurrence relation for the graph $W_n - e$, i.e. F^{-}_{n-1}.

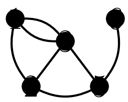
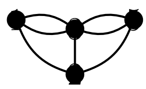
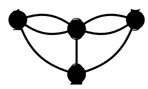


(a) F^{++}_{n-1}

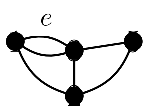
$=$ 

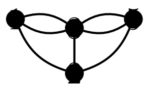
$+$ 

(b) F^{+}_{n-1} (c) $\widetilde{F}^{++}_{n-2}$

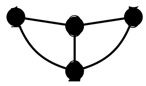
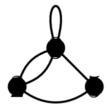
$=$  $+$  $+\, y \Big($  $\Big)$

(d) F'^{+}_{n-1} (e) F^{++}_{n-2} (f) F^{++}_{n-2}

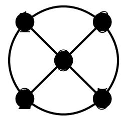
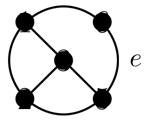
$= \; x \Big($ 

$\Big) + (1+y) \Big($  $\Big)$

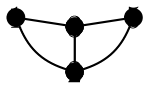
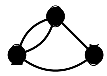
(g) F^{+}_{n-2} (h) F^{++}_{n-2}

$= \quad x\Bigg($  $+$  $\Bigg)$

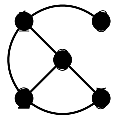
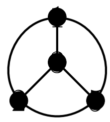
(i) F_{n-2} (j) $\widetilde{F}^{+}_{n-3}$

$+(1+y)\Bigg($  $-$  $\Bigg)$

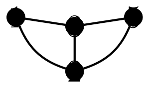
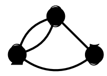
(k) W_{n-1} (l) F^{-}_{n-2}

$= \quad x\Bigg($  $\Bigg) + x\,y\Bigg($  $\Bigg) + (1+y)\,T_{W_{n-1}}(x,y)$

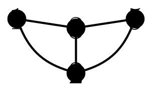
(m) F_{n-2} (n) F^{+}_{n-3}

$-(1+y)\Bigg($  $+$  $\Bigg)$

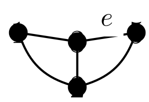
(o) F'_{n-2} (p) W_{n-2}

$= \quad x\Bigg($  $\Bigg) + x\,y\Bigg($  $\Bigg) + (1+y)\,T_{W_{n-1}}(x,y)$

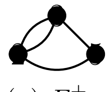
(q) F_{n-2} (r) F^{+}_{n-3}

$-(x+x\,y)\Bigg($  $\Bigg) - (1+y)\,T_{W_{n-2}}(x,y)$

(s) F_{n-2}

$= (1+y)\,T_{W_{n-1}}(x,y) - (1+y)\,T_{W_{n-2}}(x,y) - x\,y\Bigg($  $\Bigg)$

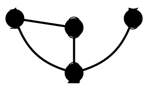
(t) F_{n-2}

$+x\,y\Bigg($  $\Bigg)$

(u) F^{+}_{n-3}

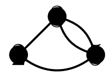
$$= (1+y)\,T_{W_{n-1}}(x,y) - (1+y)\,T_{W_{n-2}}(x,y)$$

$$-x\,y\Big($$



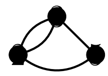
(v) F'_{n-3}

$$+$$



(w) F^{+}_{n-3}

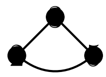
$$\Big) + x\,y\Big($$



(x) F^{+}_{n-3}

$$\Big)$$

$$= (1+y)\,T_{W_{n-1}}(x,y) - (1+y)\,T_{W_{n-2}}(x,y) - x^2\,y\Big($$



(y) F_{n-3}

$$\Big) \tag{14}$$

Figure 18: Tutte polynomial recurrence relation for the graph W_n/e, i.e. F^{++}_{n-1}

Therefore, the linear recurrence used to calculate the Tutte polynomial for the class of wheel graphs is the costructed by the addition of Equation (13) and Equation (14) as

$$T_{W_n}(x,y) = (x+y+2)\,T_{W_{n-1}}(x,y) - (x+xy+y+1)\,T_{W_{n-2}}(x,y) + x\,y\,T_{W_{n-3}}(x,y). \tag{15}$$

From Equation (15), we get the characteristic polynomial,

$$\lambda^3 - (x+y+2)\lambda^2 + (x+x\,y+y+1)\lambda - xy = 0 \tag{16}$$

for which the roots, λ_1, λ_2, and λ_3, can be solved. It is easy to see that λ_3 is equal to 1 by factoring the term $(\lambda - 1)$ from Equation (16) as the following,

$$(\lambda^2 - (x+y+1)\lambda + x\,y)(\lambda - 1) = 0. \tag{17}$$

Looking at Equation (17), it is interesting that the first trinomial is exactly that of Equation (10) for the fan graph linear recurrence previously discussed. This result is not simply by accident. In fact, it represents the single edge whose addition to a fan graph creates a wheel graph. Thus, we have that the exact solutions for λ_1 and λ_2 are

$$\frac{x+y+1 \pm r}{2}, \quad \text{when } r = \sqrt{(x+y+1)^2 - 4xy}.$$

Now that the three roots of this recurrence relation have been determined, the Tutte polynomials of the graphs W_1, W_2, and W_3 can be used as the initial conditions, and the particular solution can be obtained as follows,

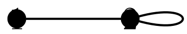
$$T_{W_n}(x,y) = \left(\frac{x+y+1+r}{2}\right)^n + \left(\frac{x+y+1-r}{2}\right)^n + xy - x - y - 1 \tag{18}$$

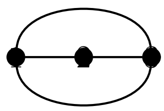
for all $n \geq 1$.

Now, we demonstrate Equation (18) on the wheel graph with 4 spokes.

$$\begin{aligned} T_{W_4}(x,y) &= \left(\frac{x+y+1+r}{2}\right)^4 + \left(\frac{x+y+1-r}{2}\right)^4 + xy - x - y - 1 \\ &= 3\,x + 6\,x^2 + 4\,x^3 + x^4 + 3\,y + 9\,x\,y + 4\,x^2\,y + 6\,y^2 + 4\,x\,y^2 + 4\,y^3 + y^4. \end{aligned} \tag{19}$$

The first paper to present this explicit formula for the family of wheel graphs was "Recursive Families of Graphs" written by N. L. Biggs, R. M. Damerell, and D. A. Sands and published in 1971. Equivalent recreations of Equation (18) can also be found in [11] as well as [5]. Knowledge of the class of wheel graphs and their Tutte polynomials are needed extensively through the calculation of the Tutte polynomial formula for the class of twisted wheel graphs that follows.

$T_{W_1}(x,y)$ =  = xy

$T_{W_2}(x,y)$ =  = $x^2 + x + xy + y + y^2$

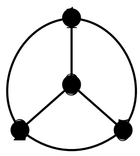
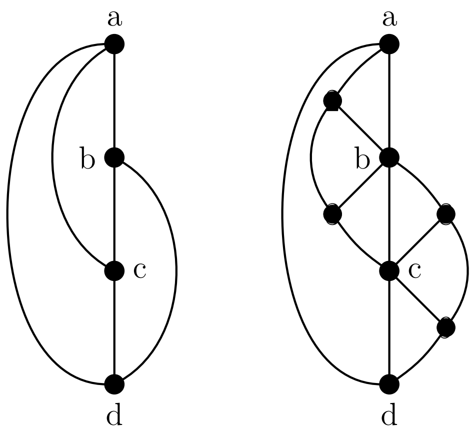
$T_{W_3}(x,y)$ =  = $x^3 + 3x^2 + 2x + 4xy + 2y + 3y^2 + y^3$

Figure 19: First three Tutte polynomial terms from the sequence $\{W_n\}$.

4.3 Twisted Wheel Graphs

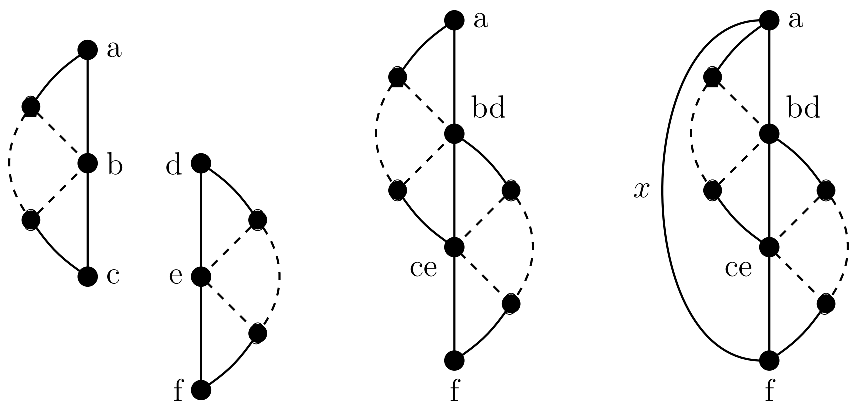
The *twisted wheel graph*, denoted as $TW_{n,m}$, can be constructed from the complete graph on 4 vertices, K_4. In Figure 20a, we show the graphical representation of K_4 that will be used to create $TW_{n,m}$ and define $V(K_4) = \{a, b, c, d\}$. Proceed by subdividing the edges $a\,c$ and $b\,d$ by the addition of one or more vertices. Then each new vertex on edge $a\,c$ is joined to vertex b. Similarly, each new vertex on edge $b\,d$ is joined to vertex c. This is depicted in Figure 20b. This algorithm used to build the $TW_{n,m}$ graph was taken from [13].



(a) K_4. (b) $TW_{n,m}$.

Figure 20: The construction of $TW_{n,m}$ from K_4.

It is obvious that the twisted wheel graph can also be formed by the parallel connection (defined below) across two fan graphs, F_n and F_m, plus the addition of one edge increasing the vertices of degree 2 to degree 3. We will refer to this additional edge as $x \in E(TW_{n,m})$ throughout the rest of this paper and define parallel connection in Section 4.3.1. As seen in Figure 21, the hub of F_n is adjacent to the hub of F_m; i.e. the hub of one fan graph is a rim vertex of the other fan graph.



(a) F_n and F_m (b) $P(F_n, F_m)$ (c) $TW_{n,m}$

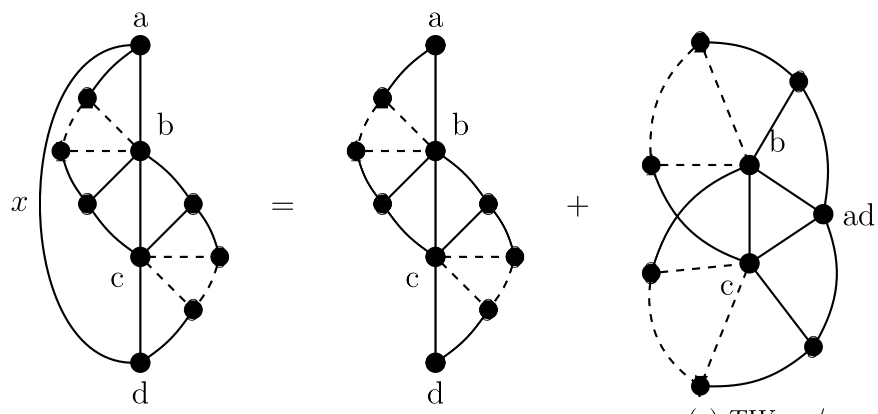
Figure 21: $TW_{n,m}$.

On my first attempt to solve for the Tutte polynomial formula of the class of twisted wheel graphs, I exercised a similar technique of deletion and contraction reductions with algebraic substitutions based on the respective general assumptions as performed in both the fan graph and wheel graph Tutte polynomial recurrence relations. However, this rather quickly turned into a graphical mess. Upon further review, it was suggested that the two graph minors formed after one deletion and contraction cycle on a specific edge might yield a helpful set of graph minors. This set of deletion and contraction operations had to be performed upon edge x. The graph minor $TW_{n,m} - x$ is a parallel connection across the graphs F_n and F_m. While the graph minor $TW_{n,m}/x$ is a generalized parallel connection of W_n and W_m across K_3. In graph theory, the generalized parallel connection of two graphs across a complete graph of three vertices is also known as a *3-clique sum*. It is important to note that F_n and W_n are graphs on the same number of spokes, n. Similarly, F_m and W_m are graphs on m spokes. That is, no spokes are lost in the two reduction operations.

Now, let $TW_{n,m}$ be a twisted wheel graph and $x \in E(TW_{n,m})$ such that x is the depicted edge in Figure 21c and Figure 22a. It follows Definition 3.1.1 of the Tutte polynomial that

$$T_{TW_{n,m}}(x, y) = T_{TW_{n,m}-x}(x, y) + T_{TW_{n,m}/x}(x, y) \tag{20}$$

Thus the Tutte polynomial of the twisted wheel graph follows the recurrence relation displayed in Figure 22.



(a) $TW_{n,m}$ (b) $TW_{n,m} - x$ (c) $TW_{n,m}/x$

Figure 22: Graphical Representation of $TW_{n,m}$ after the deletion and contraction of the edge x.

The following two sections of this paper focus on these two graph minors, $TW_{n,m} - x$ and $TW_{n,m}/x$, which satisfy the definitions of the parallel connection and the generalized parallel connection, respectively. Our goal for these sections is to show how these two graph minors follow the splitting formulas of [1] and serve as the building blocks for the formula for the Tutte polynomial of the class of twisted wheel graphs.

Andrzejak presents his paper in matroid language, which we will change to graph language for our purposes.

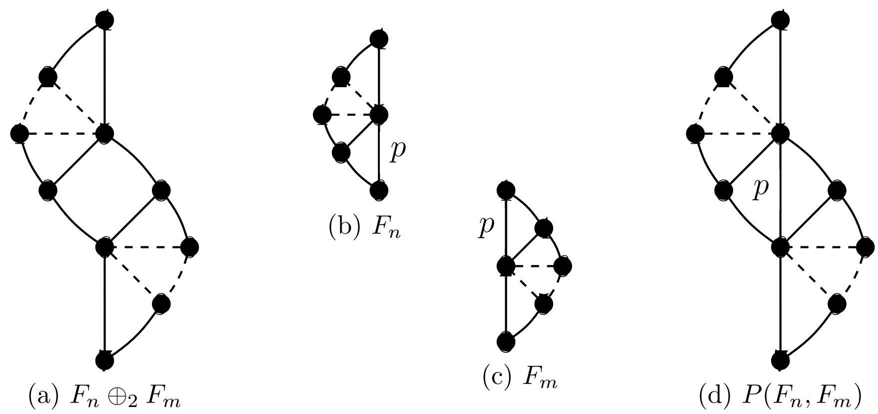
4.3.1 Parallel Connection

A *splitting formula* is an arithmetic rule that tells how to find the Tutte polynomial of a graph from the Tutte polynomials of its graph minors that are typically smaller and simpler to calculate [1].

Before defining a parallel connection and its splitting formula, we define a *2-sum*, denoted as $G_1 \oplus_2 G_2$. This definition and the definition of a parallel sum are derived from [1] and [13]. Let G_1 and G_2 be graphs with $E(G_1) \cap E(G_2) = \{p\}$ and $|E(G_1)|, |E(G_2)| \geq 3$. If the edge p is neither a loop nor a bridge in G_1 or G_2, then $G_1 \oplus_2 G_2$ is the graph on $E(G_1) \cup E(G_2) = \{p\}$. That is, a 2-sum across the edge p is created by identifying the edge p of G_1 and G_2 followed by the subsequent removal of edge p [13]. It is important to note that p is equivalent to the K_2 graph, i.e. the complete graph on two vertices.

The definition of *parallel connection* is very similar to the 2-sum. Let G_1 and G_2 be graphs and $E(G_1) \cap E(G_2) = \{p\}$. If edge p is neither a loop nor an isthmus in G_1 or G_2 and $|E(G_1)|, |E(G_2)| \geq 3$, then the parallel connection of G_1 and G_2, denoted as $P(G_1, G_2)$, is the graph on $E(G_1) \cup E(G_2)$ in which the edges p are identified. The important connection between the 2-sum and the parallel connection is that the $G_1 \oplus_2 G_2$ of G_1 and G_2 is in fact $P(G_1, G_2) - p$ [1].

Our graph, $TW_{n,m} - x$, is exactly a parallel connection, see Figure 23d. That is, G_1 and G_2 are the fan graphs F_n and F_m for which $E(F_n) \cap E(F_m) = \{p\}$.



(a) $F_n \oplus_2 F_m$ (b) F_n (c) F_m (d) $P(F_n, F_m)$

Figure 23: 2-Sum and Parallel Connection of F_n and F_m.

In [1], Andrzejak gives the splitting formula for the Tutte polynomial of a parallel connection as follows,

$$T_{P(G_1,G_2)}(x,y) = (xy-x-y)^{-1} \begin{bmatrix} T_{G_1/p} & T_{G_1-p} \end{bmatrix} \begin{bmatrix} xy-y-1 & -1 \\ -1 & y-1 \end{bmatrix} \begin{bmatrix} T_{G_2/p} \\ T_{G_2-p} \end{bmatrix}. \quad (21)$$

Let $G = TW_{n,m} - x$ with $n, m \in \{2, 3, 4, \ldots\}$ be the graph of a parallel connection in Figure 23d . Thus it follows the definition of parallel connection that F_n and F_m can be substituted for G_1 and G_2, respectively. Hence $P(G_1, G_2)$ can be rewritten as $P(F_n, F_m)$. Then we apply Equation (21) to $P(F_n, F_m)$ with the following substitutions

- $T_{G_1/p}(x,y) = T_{F_n/p}(x,y) = T_{F_n}(x,y) - xTF_{n-1}(x,y)$;
- $T_{G_1-p}(x,y) = T_{F_n-p}(x,y) = xT_{F_{n-1}}(x,y)$;
- $T_{G_2/p}(x,y) = T_{F_m/p}(x,y) = T_{F_m}(x,y) - xTF_{m-1}(x,y)$;
- $T_{G_2-p}(x,y) = T_{F_m-p}(x,y) = xT_{F_{m-1}}(x,y)$.

Thus the Tutte polynomial formula for the sequence of graphs $P(F_n, F_m)$ is

$$T_{P(F_n,F_m)}(x,y) = (xy-x-y)^{-1} \begin{bmatrix} T_{F_n}(x,y) - xTF_{n-1}(x,y) & xT_{F_{n-1}}(x,y) \end{bmatrix} \times \times \begin{bmatrix} xy-y-1 & -1 \\ -1 & y-1 \end{bmatrix} \begin{bmatrix} T_{F_m}(x,y) - xTF_{m-1}(x,y) \\ xT_{F_{m-1}}(x,y) \end{bmatrix}. \quad (22)$$

for $n, m \geq 2$.

Equation (22) is the first piece to the recursion model for the Tutte polynomial for the class of twisted wheel graphs.

Now, we present an example of the Equation (22) in which we compute the Tutte polynomial of the graph $TW_{4,4} - x$, i.e. the graph $P(F_n, F_m)$.

$$\begin{aligned} T_{TW_{4,4}-x}(x,y) = {} & x + 6\,x^2 + 15\,x^3 + 20\,x^4 + 15\,x^5 + 6\,x^6 + x^7 + y + 10\,x\,y + 30\,x^2\,y \\ & + 40\,x^3\,y + 25\,x^4\,y + 6\,x^5\,y + 5\,y^2 + 26\,x\,y^2 + 42\,x^2\,y^2 + 26\,x^3\,y^2 \\ & + 5\,x^4\,y^2 + 10\,y^3 + 28\,x\,y^3 + 21\,x^2\,y^3 + 4\,x^3\,y^3 + 10\,y^4 + 13\,x^2\,y^4 \\ & + 3\,x^2\,y^4 + 5\,y^5 + 2\,x\,y^5 + y^6. \end{aligned} \quad (23)$$

This example shows how complex these calculations can be even with n and m at trivial values. Furthermore, $T_{TW_{4,4}}(x,y)$ stresses the importance of the Tutte polynomial formula for the class of twisted wheel graphs because, remember, this computation is merely half of the work necessary to solve for $T_{TW_{4,4}}(x,y)$.

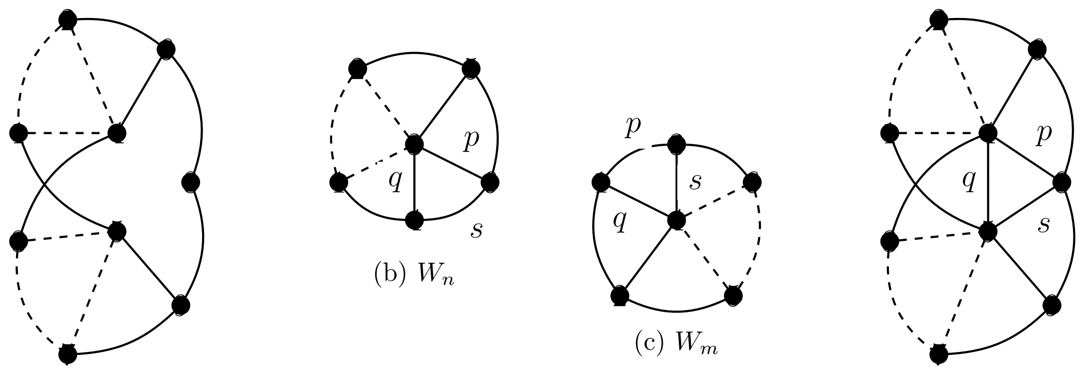
4.3.2 General Parallel Connection Across K_3

A *generalized parallel connection* (GPC) of two graphs G_1 and G_2, whose definition is taken from [1], is a graph on $E(G_1) \cup E(G_2)$. In a GPC, we say that N is the *connecting minor* across which the graph is based. That is, N is an induced subgraph of G, and $E(N) = E(G_1) \cap E(G_2)$. Thus, if $|E(G_1)|, |E(G_2)| \geq 7$, then the GPC across G_1 and G_2, denoted as $P_N(G_1, G_2)$, is the identification of the edges of N. A GPC may more easily be described by imagining that you simply slide G_1 and G_2 upon one another by matching the edges of N.

Similar to the relationship between the 2-sum and parallel connection is the relationship between the 3-sum and the GPC when N is a complete graph of three vertices; that is a K_3 or a triangle. A *3-sum* of two graphs G_1 and G_2, denoted as $G_1 \oplus_3 G_2$, is simply $P_N(G_1, G_2) - E(N)$ [1], that is, the identification of the edges of N and the subsequent removal of $E(N)$. Andrzejak's paper gives the splitting formula for both of these graphs, but we will focus on the prior.

In this section, we study the minor $TW_{n,m}/x$, which is a GPC where $G_1 = W_n$ and $G_2 = W_m$. Thus the connecting minor, N, is a complete graph of order 3, K_3, or a triangle with $E(N) = E(W_n) \cap E(W_m) = \{p, q, s\}$. Since W_n and W_m must contain more than six edges, we have that $n, m \geq 4$, i.e. each wheel graph must be of at least order 5. To ensure that a $TW_{n,m}/x$ cannot be represented as a parallel connection, one more requirement is defined. For the $E(N) = \{p, q, s\}$, in W_n there must be a 3-cycle $C_n \cup \{q\}$ with $C_n \subseteq E(W_n) - E(N)$ and a 3-cycle $C_m \cup \{p\}$ with $C_m \subseteq E(W_m) - E(N)$ [1].

For an picture of these graphs, reference Figure 24. It is important to note that n and m will be the same value for n and m we used in the splitting formula for the parallel connection on F_n and F_m in the previous section.



(a) $W_n \oplus_3 W_m$ (b) W_n (c) W_m (d) $P_N(W_n, W_m)$

Figure 24: 3-Sum and Generalized Parallel Connection of W_n and W_m.

Equivalent variations of the splitting formula for the Tutte polynomial of a

GPC can be found in a few sources such as [4]. We will continue to use [1] for consistency, but each variation should give equivalent polynomial results for the Tutte polynomial calculations. Andrzejak substitutes

$$A_1 = T_{G_1/q} + T_{G_1/p} + T_{G_1/s} \tag{24}$$

and

$$A_2 = T_{G_2/q} + T_{G_2/p} + T_{G_2/s} \tag{25}$$

to simplify the splitting formula for the GPC of graph G_1 and G_2 with the K_3 connecting minor. The splitting formula for the Tutte polynomial of a GPC across a K_3 is given by Andrezejak as follows,

$$\begin{aligned} &T_{P_N(G_1,G_2)}(x,y) \\ &\quad = (x\,y - x - y)^{-1}(x\,y - x - y - 1)^{-1}((x\,y - x - y - 1)\,y \\ &\qquad \times [T_{G_1/q}(x,y)\,T_{G_2/q}(x,y) + T_{G_1/p}(x,y)\,T_{G_2/p}(x,y) + T_{G_1/s}(x,y)\,T_{G_2/s}(x,y)] \\ &\qquad + 2\,y^3\,[T_{G_1}(x,y)\,T_{G_2/q/p/s}(x,y) + T_{G_2}(x,y)\,T_{G_1/q/p/s}(x,y)] + y^2\,A_1\,A_2 \\ &\qquad + y\,(1-y)\,[T_{G_1}(x,y)\,A_2 + T_{G_2}(x,y)\,A_1)] \\ &\qquad - y^3\,(1+x)\,[T_{G_1/q/p/s}(x,y)\,A_2 + T_{G_2/q/p/s}(x,y)\,A_1] \\ &\qquad + y^3\,(x^2 + x + y + 3\,x\,y)\,T_{G_1/q/p/s}(x,y)\,T_{G_2/q/p/s}(x,y) \\ &\qquad + (y - 1)^2\,T_{G_1}(x,y)\,T_{G_2}(x,y)). \end{aligned} \tag{26}$$

Let $G = TW_{n,m}/x$ for $n, m \in \{4, 5, 6, \ldots\}$ be a GPC across K_3 as seen in Figure 24d. Thus N is a K_3 graph and in substitution for G_1 and G_2, we have W_n and W_m, respectively. Hence we can rewrite $P_N(G_1, G_2)$ as $P_N(W_n, W_m)$. Thus using Equation (26), we are able to evaluate the Tutte polynomial for the GPC across K_3 for our graph $TW_{n,m}/x$ with the following substitutions

- $A_1 = T_{G_1/q} + T_{G_1/p} + T_{G_1/s} = (T_{W_n}(x,y) - x\,T_{F_{n-1}}(x,y) - T_{W_{n-1}}(x,y))$ $+(T_{W_n}(x,y) - x\,T_{F_{n-1}}(x,y) - T_{W_{n-1}}(x,y)) + (T_{W_n}(x,y) - T_{F_n}(x,y))$;
- $A_2 = T_{G_2/q} + T_{G_2/p} + T_{G_2/s} = (T_{W_m}(x,y) - x\,T_{F_{m-1}}(x,y) - T_{W_{m-1}}(x,y))$ $+(T_{W_m}(x,y) - T_{F_m}(x,y)) + (T_{W_m}(x,y) - x\,T_{F_{m-1}}(x,y) - T_{W_{m-1}}(x,y))$;
- $T_{G_1/q}(x,y) = T_{W_n/q}(x,y) = T_{W_n}(x,y) - x\,T_{F_{n-1}}(x,y) - T_{W_{n-1}}(x,y)$;
- $T_{G_1/p}(x,y) = T_{W_n/p}(x,y) = T_{W_n}(x,y) - x\,T_{F_{n-1}}(x,y) - T_{W_{n-1}}(x,y)$;
- $T_{G_1/s}(x,y) = T_{W_1/s} = T_{W_n}(x,y) - T_{F_n}(x,y)$;
- $T_{G_2/q}(x,y) = T_{W_m/q}(x,y) = T_{W_m}(x,y) - x\,T_{F_{m-1}}(x,y) - T_{W_{m-1}}(x,y)$;
- $T_{G_2/p}(x,y) = T_{W_m/p} = T_{W_m}(x,y) - T_{F_m}(x,y)$
- $T_{G_2/s}(x,y) = T_{W_m/s}(x,y) = T_{W_m}(x,y) - x\,T_{F_{m-1}}(x,y) - T_{W_{m-1}}(x,y)$;
- $T_{G_1/q/p/s}(x,y) = T_{W_n/q/p/s}(x,y) = T_{W_{n-1}}(x,y) - x\,T_{F_{n-2}}(x,y) - T_{W_{n-2}}(x,y)$;

- $T_{G_2/q/p/s}(x,y) = T_{W_m/q/p/s}(x,y) = T_{W_{m-1}}(x,y) - x\,T_{F_{m-2}}(x,y) - T_{W_{m-2}}(x,y)$.

Thus for continued simplification, we have

$$\begin{aligned} A_1 &= (T_{W_n}(x,y) - x\,T_{F_{n-1}}(x,y) - T_{W_{n-1}}(x,y)) + (T_{W_n}(x,y) - x\,T_{F_{n-1}}(x,y) \\ &\quad - T_{W_{n-1}}(x,y)) + (T_{W_n}(x,y) - T_{F_n}(x,y)) \end{aligned} \tag{27}$$

and

$$\begin{aligned} A_2 &= (T_{W_m}(x,y) - x\,T_{F_{m-1}}(x,y) - T_{W_{m-1}}(x,y)) + (T_{W_m}(x,y) - T_{F_m}(x,y)) \\ &\quad + (T_{W_m}(x,y) - x\,T_{F_{m-1}}(x,y) - T_{W_{m-1}}(x,y)) \end{aligned} \tag{28}$$

as the new values of A_1 and A_2 for $TW_{n,m}/x$. Hence we claim that the formula for the Tutte polynomial of $TW_{n,m}/x$ after the as follows,

$$\begin{aligned} &T_{P_N(W_n,W_m)}(x,y) \\ &\quad = (x\,y - x - y)^{-1}(x\,y - x - y - 1)^{-1}((x\,y - x - y - 1)\,y \\ &\quad\times[(T_{W_n}(x,y) - xT_{F_{n-1}}(x,y) - T_{W_{n-1}}(x,y))(T_{W_m}(x,y) - xT_{F_{m-1}}(x,y) - T_{W_{m-1}}(x,y)) \\ &\quad + (T_{W_n}(x,y) - x\,T_{F_{n-1}}(x,y) - T_{W_{n-1}}(x,y))(T_{W_m}(x,y) - T_{F_m}(x,y)) \\ &\quad + (T_{W_n}(x,y) - T_{F_n}(x,y))(T_{W_m}(x,y) - x\,T_{F_{m-1}}(x,y) - T_{W_{m-1}}(x,y))] \\ &\quad +2\,y^3\,[T_{W_n}(x,y)\,(T_{W_{m-1}}(x,y) - x\,T_{F_{m-2}}(x,y) - T_{W_{m-2}}(x,y)) + T_{W_m}(x,y)\,(T_{W_{n-1}}(x,y) \\ &\quad - x\,T_{F_{n-2}}(x,y) - T_{W_{n-2}}(x,y))] + y^2\,A_1\,A_2 + y\,(1-y)\,[T_{W_n}(x,y)\,A_2 + T_{W_m}(x,y)\,A_1)] \\ &\quad - y^3\,(1+x)\,[(T_{W_{n-1}}(x,y) - x\,T_{F_{n-2}}(x,y) - T_{W_{n-2}}(x,y))\,A_2 + (T_{W_{m-1}}(x,y) \\ &\quad - xT_{F_{m-2}}(x,y) - T_{W_{m-2}}(x,y))\,A_1] + y^3\,(x^2\,+\,x\,+\,y\,+\,3\,x\,y)\,(T_{W_{n-1}}(x,y) \\ &\quad - x\,T_{F_{n-2}}(x,y) - T_{W_{n-2}}(x,y)) \times (T_{W_{m-1}}(x,y) - x\,T_{F_{m-2}}(x,y) - T_{W_{m-2}}(x,y)) \\ &\quad + (y-1)^2\,T_{W_n}(x,y)\,T_{W_m}(x,y)) \end{aligned} \tag{29}$$

for $n, m \geq 4$.

A calculating example of Equation (29) is now presented on the graph $P_N(W_4, W_4)$;

that is, $T_{TW_{4,4}/x}(x,y)$.

$$T_{TW_{4,4}/x}(x,y) = 9\,x+27\,x^2+33\,x^3+21\,x^4+7\,x^5+x^6+9\,y+46\,xy+63\,x^2y+35\,x^3y \\ +7\,x^4y+28\,y^2+67\,xy^2+44\,x^2y^2+9\,x^3y^2+38\,y^3+47\,xy^3+14\,x^2y^3 \\ +31\,y^4+19\,xy^4+2\,x^2y^4+17\,y^5+4\,xy^5+6\,y^6+y^7. \tag{30}$$

Thus, $T_{TW_{4,4}}(x,y)$ equals the sum of Equation (23) and Equation (30). That is,

$$\begin{aligned} T_{TW_{4,4}}(x,y) = &\Big(x+6\,x^2+15\,x^3+20\,x^4+15\,x^5+6\,x^6+x^7+y+10\,x\,y+30\,x^2\,y \\ &+40\,x^3\,y+25\,x^4\,y+6\,x^5\,y+5\,y^2+26\,x\,y^2+42\,x^2\,y^2+26\,x^3\,y^2 \\ &+5\,x^4\,y^2+10\,y^3+28\,x\,y^3+21\,x^2\,y^3+4\,x^3\,y^3+10\,y^4+13\,x^2\,y^4 \\ &+3\,x^2\,y^4+5\,y^5+2\,x\,y^5+y^6\Big)+\Big(9\,x+27\,x^2+33\,x^3+21\,x^4 \\ &+x^6+9\,y+46\,xy+63\,x^2y+35\,x^3y+7\,x^4y+28\,y^2+67\,xy^2 \\ &+44\,x^2y^2+9\,x^3y^2+38\,y^3+47\,xy^3+14\,x^2y^3+31\,y^4+19\,xy^4 \\ &+2\,x^2y^4+17\,y^5+4\,xy^5+6\,y^6+y^7\Big) \\ = &\,10\,x+33\,x^2+48\,x^3+41\,x^4+22\,x^5+7\,x^6+x^7+10\,y+56\,x\,y+93\,x^2\,y \\ &+75\,x^3\,y+32\,x^4\,y+6\,x^5\,y+33\,y^2+93\,x\,y^2+86\,x^2\,y^2+35\,x^3\,y^2+5\,x^4\,y^2 \\ &+48\,y^3+75\,x\,y^3+35\,x^2\,y^3+4\,x^3\,y^3+41\,y^4+32\,x\,y^4+5\,x^2\,y^4+22\,y^5 \\ &+6\,x\,y^5+7\,y^6+y^7 \end{aligned} \tag{31}$$

As mentioned in Theorem 1 of Section 1 of this paper, the Tutte polynomial of a connected graph G has various capacities that can be used to count graph invariants when x and y are substituted with specific values. For example, the number of subsets of edges of a graph G can be calculated when the Tutte polynomial of G is normalized to $x = 2$ and $y = 2$. The evaluation of this graph invariant at $T_G(2,2)$ equals $2^{|E(G)|}$, which is also a known way to calculate the number os subsets of edges of a graph. The following is a calculating example this specific counting capacity.

$$\begin{aligned} T_{TW_{4,4}}(2,2) &= 10\,(2){+}33\,(2)^2{+}48\,(2)^3{+}41\,(2)^4{+}22\,(2)^5{+}7\,(2)^6{+}x^7{+}10\,y{+}56\,x\,y \\ &\quad +93\,x^2\,y{+}75\,(2)^3\,(2){+}32\,(2)^4\,(2){+}6\,(2)^5\,(2){+}33\,(2)^2{+}93\,(2)\,(2)^2 \\ &\quad +86\,(2)^2\,(2)^2{+}35\,(2)^3\,(2)^2{+}5\,(2)^4\,(2)^2{+}48\,(2)^3{+}75\,(2)\,(2)^3{+}35\,(2)^2\,(2)^3 \\ &\quad +4\,(2)^3\,(2)^3\,{+}41\,(2)^4{+}32\,(2)\,(2)^4{+}5\,(2)^2\,(2)^4{+}22\,(2)^5{+}6\,(2)\,(2)^5 \\ &\quad +7\,(2)^6+(2)^7 \\ &= 16,384 \\ &= 2^{14} \\ &= 2^{|E(TW_{4,4})|}. \end{aligned} \tag{32}$$

This equality supports the assumptions of Theorem 1 from Section 1. Other ways of applying this equation could be through the study of additional known invariant normalizations in comparison to the evaluations of the Tutte polynomial formula for $TW_{n,m}/x$.

5 Further Study

The computation of the Tutte polynomial of specific members from the class of twisted wheel graphs quickly becomes complex as the number of spokes, n and m, increase. For example,

$$\begin{aligned} T_{TW_{5,5}}(x,y) = {} & 17\,x+72\,x^2+144\,x^3+180\,x^4+154\,x^5+92\,x^6+37\,x^7+9\,x^8+x^9+17\,y \\ & + 127\,xy + 315\,x^2y + 423\,x^3y + 353\,x^4y + 187x^5y + 58\,x^6y + 8\,x^7y \\ & + 72\,y^2 + 315\,xy^2 + 531x^2y^2 + 481\,x^3y^2 + 250\,x^4y^2 + 68\,x^5y^2 + 7\,x^6y^2 \\ & + 144\,y^3 + 423\,xy^3 + 481\,x^2y^3 + 271\,x^3y^3 + 71\,x^4y^3 + 6\,x^5y3 + 180\,y^4 \\ & + 353\,xy^4 + 250\,x^2y^4 + 71x^3y^4 + 5x^4y^4 + 154\,y^5 + 187\,xy^5 + 69\,x^2y^5 \\ & + 6\,x^3y^5 + 92\,y^6 + 58\,xy^6 + 7\,x^2 + y^6 + 37\,y^7 + 8\,xy^7 + 9\,y^8 + y^9 \end{aligned} \tag{33}$$

compared to

$$\begin{aligned} T_{TW_{5,6}}(x,y) = {} & 21\,x+99\,x^2+226\,x^3+329\,x^4+335\,x^5+246\,x^6+129\,x^7+46\,x^8+10\,x^9 \\ & +x^{10}+21\,y+177\,xy+514\,x^2y+832\,x^3y+865\,x^4y+602\,x^5y+274\,x^6y \\ & +74\,x^7y+9\,x^8y+99\,y^2+514\,xy^2+1071\,x^2y^2+1249\,x^3y^2+829\,x^4y^2 \\ & +384\,x^5y^2+89\,x^6y^2+8\,x^7y^2+226\,y^3+832\,xy^3+1249\,x^2y^3+1001\,x^3y^3 \\ & +440\,x^4y^3+95\,x^5y^3+x\,x^6y^3+329\,y^4+865\,xy^4+892\,x^2y^4+440\,x^3y^4 \\ & +96\,x^4y^4+6\,x^5y^4+335\,y^5+602\,xy^5+384\,x^2y^5+95\,x^3y^5+6\,x^4y^5 \\ & +246\,y^6+274\,xy^6+89\,x^2y^6+7\,x^3y^6+129\,y^7+74\,xy^7+8x^2y^7+46\,y^8 \\ & +9\,xy^8+10\,y^9+y^{10}. \end{aligned} \tag{34}$$

These examples merely reflect the addition of one spoke as m changes from 5 to 6; thus, showing the difficulty of the Tutte polynomial calculations with just pen and paper and the relevance of the Tutte polynomial formula for the class of twisted wheel graphs. Using the Equations (11) and (19), I created a Mathematica file containing the Tutte polynomials for F_i and W_i with $i \in \{1, 2, \ldots, 10\}$. This file helped throughout initial calculations but only addressed the graphs with a finite number of spokes. Therefore, it would be interesting to conduct further study focused on the creation of a computer program that could calculate the Tutte polynomial of a twisted wheel graph for any number of spokes, $n, m \geq 4$. Interestingly, there does exist an code for the computer program Sage that has the capacity to calculate the Tutte polynomial of certain graphs and matroids. This algorithm identifies edges as loops, as cut edges, or as neither a loop or a cut edge. Then it either deletes, contracts, or both deletes and contracts the edge according to its classification and returns the Tutte polynomial of the graph of interest in its expanded form. If the graph is not an archived in Sage's memory, then it is helpful to rely upon knowledge of the matrix representation of the cycle matroid of the graph. More information about matroid theory can be found in [14].

Another area of interest focused on the Tutte polynomial of the class of twisted wheel graph would be an investigation of *Tutte uniqueness*. First, two graphs G and H are *Tutte equivalent* if they share the same Tutte polynomial [8]. Then, a graph G is *Tutte unique* if every Tutte equivalent graph H is isomorphic to G [6]. That is, Tutte unique graphs are distinguishable by their Tutte polynomial. Also, a class of graphs can be called *Tutte unique* if any two graphs G and H of that class have different polynomials [8]. A simple example of a class of graphs that are Tutte unique is the class of cycle graphs. In contrast, the class of trees across n vertices, T_n, whose Tutte polynomial is $T_{T_n}(x, y) = x^n$ is a class that is not Tutte unique [8]. A paper published in 2009 proved that like cycles, wheel, and fans, among other families of graphs, the class of twisted wheel graphs were also Tutte unique [6]. It would be interesting to attempt to reconstruct this study to gain experience in the mechanics of Tutte uniqueness.

In addition to Tutte uniqueness, it would be interesting to study the effect of rooted vertices upon the Tutte polynomials of the class of twisted wheel graphs. A *root vertex* is a distinguished vertex, and a *rooted graph* is a graph with a root vertex [8]. Gary Gordon begins a discussion about the effect a root vertex has on the Tutte polynomial of non-trivial classes of graphs, which he notes as important in communication theory, in [8]. In comparison to Theorem 1 of this paper's Section 1 and Theorem 3.5 of [8], Gordon proposes changes to these counting capacities to allow G to be not only a graph but more strictly defined as a rooted graph. He includes details on subsets, spanning trees, spanning sets, rooted subtrees, and acyclic orientations with unique source v and then poses the challenge of extending all of the other evaluations of the Tutte polynomial to include a rooted vertex. Also, he poses the open question, is it true that any two rooted graphs have distinct Tutte polynomials? This would be an interesting question to study in respect the Tutte polynomial of the class of twisted wheel graphs.

A class of graphs is called *recursive* if the Tutte polynomials of its members satisfy a linear recurrence relation. Just as the families of fan graphs and wheel graphs are recursive, so is the class of twisted wheel graphs. If a family of graphs can be built from a given initial graph by means of a repeated set of elementary operations involving, then it is called a *recursively constructible* [12]. The elementary operations can include the addition of a new set of vertices, the addition of a new set of edges, and the deletion of a fixed set of edges. It is known that the class of wheels graphs is recursively constructible [12], thus an interesting area of study could aim to answer a question, is the family of twisted wheel graphs recursively constructible? And if the answer is yes, what is the collection of elementary operations used to build successive members of the class of twisted wheel graphs? Also, since we claim that the class of twisted wheel graphs is recursive, a good question to ask would be, can we find a formula for the Tutte polynomial of the twisted wheel graphs, as we did with the classes of fan graphs and wheel graphs? The computation of a closed form for the twisted wheels should be possible which brings to mind, what are the necessary basic relationships used in this situation? Once those are determined, a construction pattern similar to those used in Section 4.1 and Section 4.2 should be practiced in this Tutte polynomial formula calculation in search of a linear homogenous linear recurrence.

Lastly, it would also be interesting to move into calculations for the Tutte polynomial of other classes of graphs that have not yet been computed. This type of further study would serve a similar purpose as this paper as an addition to collections of Tutte polynomial formulas such as [11] and to prompt further study into computer programs, Tutte uniqueness, the effects of rooted vertices, the classification of recursively constructibility, and other applications of graph theory.

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